The contact process on dynamic graphs

Mini-course for the Summer School
Particle systems in random environments

Daniel Valesin*

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Abstract

In this lecture series, we study recent developments on the contact process on evolving graphs. The contact process is a simple model for the spread of an infection in a population. In the last two decades, there has been growing interest on the behavior of this process on random graph models which reflect real-world populations. An extra layer of realism and mathematical complexity comes from allowing the graph to evolve simultaneously with the infection, and a rapidly growing body of work has been dedicated to this sort of model. After giving a brief overview of the theory of the classical contact process, we focus on two recent developments on dynamical graphs. The first of these is the study of the contact process on dynamical percolation on the Euclidean lattice. This is an appealing extension of the classical contact process, with many interesting phenomena arise, for instance the occurrence of parameter regimes where the graph is immune to the infection, no matter how high the infection rate. The second is the contact process on dynamic d-regular graphs with an edge-switching dynamics. The highlight of this setting is a strict monotonicity result, that shows that the graph dynamics strictly helps the infection to spread.

1 Introduction

1.1 Background and motivation

The *contact process* is a class of interacting particle systems introduced by Ted Harris [16] in 1974. It is usually taken as a model for the spread of an infection in a population. In mathematical epidemiology, similar *compartmental models* (that is, models where a population is split into "compartments", such as "susceptible" and "infected") have been studied for much longer, with an important early reference being [21].

*University of Warwick, daniel.valesin@warwick.ac.uk

The dynamics of the contact process can be briefly described as follows. At any point in (continuous) time, vertices of a graph are either healthy or infected. Infected vertices become healthy spontaneously with rate one. Additionally, infected vertices transmit the infection to each of their neighbors with rate $\lambda > 0$.

Interest in the contact process is justified by its mathematical tractability on the one hand, and its rich behavior on the other. It has been the basis for the study of competition models, models with sexual reproduction and maturation, vegetation models and gene regulatory networks. It has also triggered theoretical work on criticality, random walk on dynamic random environments, metastability, shape theorems, and superprocesses. This list is illustrative and far from exhausting.

In the 1970's and 1980's, focus was on the study of the phase transition of this process on the Euclidean lattice $\mathbb{Z}^d$. Already in the seminal paper [16], it was proved that there is a critical value λ_c of the infection rate, such that, for the process started from a single infection, if $\lambda < \lambda_c$, the infection eventually disappears with probability one, whereas if $\lambda > \lambda_c$, then it persists forever with positive probability. Several important properties of the subcritical ($\lambda < \lambda_c$) and supercritical ($\lambda > \lambda_c$) were established, and we will encounter some of them. An important aspect of the supercritical contact process is its prominence in the theory of *metastability*: if the process evolves on a large box of $\mathbb{Z}^d$ with $\lambda > \lambda_c$, then the infection takes a very long time to die out (exponential in the volume of the box), and while it remains active, it gives the impression of being in equilibrium. This was identified in [8], and many other references followed.

In 1990, Bezuidenhout and Grimmett [4] proved that the process also dies out when $\lambda = \lambda_c$. In doing so, they introduced an important renormalization technique that made the proof of other interesting results possible, notably the shape theorem in all dimensions. Also in 1990, due to an important work by Pemantle [29], the contact process on trees gained attention, with the highlight being the occurrence of a double phase transition (in an intermediate "weak survival" regime, the infection survives forever on the tree, but any given finite region eventually becomes permanently free from it). The contact process on Bienaymé-Galton-Watson trees was also first considered around this time [30], inaugurating the study of this process on random graphs.

This study has picked up pace in the following two decades. The key point of interest, which offers insight into real-world epidemics, is how the geometry of the graph affects the propagation of the infection. The important works [3] and [9] showed that on graphs that exhibit a power law degree distribution, the process is in a way "always supercritical": even when the infection rate is very small, the infection is sustained for a long time, due to the presence of very highly-connected vertices ("stars").

This kind of investigation is of course made much more sophisticated and interesting when one allows the graph to co-evolve with the process. Several questions then arise, notably: does the graph dynamics help or hinder the infection? Allowing for more sophisticated models where the graph dynamics may depend on the state of the infection, one can even dream of proving theorems about cost-effective ways of blocking a pandemic with partial knowledge of the network and the infections.

Although the state of the art is not quite there yet, much has already been done. Below is a brief history. The articles that appear in bold font are the ones that are included in this course.

- Broman [6] introduced a contact process in which the recovery rate at each site changes with time, governed by an underlying Markov chain, and proved important stochastic domination results about it, one of which we will encounter here (Theorem 3.7 below).

- Steif and Warfheimer [36] continued the study of this process, proving a version of the Bezuidenhout–Grimmett theorem for it.

- Remenik [31] introduced the contact process on dynamic site percolation environment, where sites shifted between blocked and unblocked states; he studied this model in $\mathbb{Z}^d$. Later, **Linker and Remenik** [25] changed the model slightly: still in $\mathbb{Z}^d$, they considered dynamic *bond* percolation, that is, edges between neighboring lattice sites open and close for transmission. **Hilário, Ungaretti, Vares and V.** [17] studied this latter process further.

- In a series of works [20, 18, 19] (subsets of) Jacob, Linker and Mörters study the contact process on evolving power law random graphs. They show for instance that in certain situations, the "always supercritical" situation described above can be destroyed by the graph dynamics, due to stars being disintegrated long before they have a chance to sustain the infection.

- In [15], Fernley, Mörters and Ortgiese introduced and studied the contact process on a dynamic random graph that is initially Erdős–Renyi, but evolves in a way that may depend on the infection. Their study focuses on an idealized process that is a sort of local limit of their dynamics, of contact processes evolving on a dynamic random forest. They show that the critical infection rate associated to the dynamic model is distinct from that associated to a non-adaptive model.

- Cardona-Tobón, Ortgiese, Seiler and Sturm [7] studied the contact process on dynamic bond percolation on Bienaymé–Galton–Watson trees. Here the edge update rate is allowed to depend on the degrees of adjacent vertices. Several behavior regimes are proved to arise.

- **Baptista da Silva, Oliveira and V.** [11] studied the contact process on a random d-regular graph with a switching bond dynamics. They proved that, in a way that can be made precise, the graph dynamics favors the infection (the critical parameter for the dynamic graph is strictly smaller than that for the static graph). This is further investigated by **Schapira and V.** [32], which prove a stronger monotonicity result.

1.2 Course contents

As evidenced by the list above, by now the contact process on dynamic graphs is too broad a topic to cover in a six-hour lecture series, unless the course turns into a rushed survey of existing results. Here we have opted to cover a few representative works only, so that there's time to carry out some proofs in detail, with the idea that being exposed to important techniques in the field is what is most useful to the audience. Another criterion in the choice of topic was to steer away from works that go too much into random graph analysis. These can quickly lead into complicated proofs of structural lemmas about graphs, which don't really have much to do with the particle system itself.

Given these considerations, the course is split into three parts:

1. a *crash course on the contact process*, including a presentation of oriented percolation and a proof of the phase transition in $\mathbb{Z}^d$;

2. a reasonably detailed exposition of the *contact process on dynamical percolation on* $\mathbb{Z}^d$, with main focus on the work by Amitai Linker and Daniel Remenik [25];

3. an again reasonably detailed exposition of the recent work by the author and Bruno Schapira [32] on the *contact process on dynamic random d-regular graphs.*

We hope that this choice of topics will serve to teach some of the basics of the field, and to instigate curiosity that can lead to future developments.

2 Crash course on the contact process

Below is a minimalistic exposition of the contact process, with the aim of introducing notation and the main highlights of the classical theory. The standard references to learn about the contact process are [24] and [23]. The reference [37] is also recommended.

2.1 Definition and graphical construction

Let $G = (V, E)$ be a graph. Fix the parameter $\lambda > 0$, called the *infection rate*. The state space for the contact process is $\{0,1\}^V$. Given a *configuration* $\xi \in \{0,1\}^V$ and $x \in V$, we say that x is *infected* in ξ if $\xi(x) = 1$, and that x is *healthy* in ξ otherwise.

It is common to abuse notation and identify $\xi \in \{0,1\}^V$ with the set $\{x \in V : \xi(x) = 1\}$. In particular, the all-healthy configuration $\xi \equiv 0$ is sometimes denoted as an empty set, $\varnothing$.

The contact process on G with infection rate λ is a continuous-time Markov process $(\xi_t)_{t \geq 0}$ on $\{0,1\}^V$. The initial state is an arbitrary element of $\{0,1\}^V$. The dynamics is informally described by the rules:

- *spontaneous recoveries:* if $\xi(x) = 1$, then the chain performs the jump $\xi \to \xi \backslash \{x\}$ with rate 1;
- *transmissions:* if $\xi(x) = 1$, $\xi(y) = 0$ and $y \sim x$, then the chain performs the jump $\xi \to \xi \cup \{y\}$ with rate λ.

The formal construction of a process corresponding to these rules can be done via a Poisson graphical construction or via an infinitesimal pre-generator (which produces a Markov semi-group with the Hile-Yosida Theorem). The former approach has the advantage of being incomparably easier, more intuitive, useful for coupling and duality, and essentially all of the theory of the contact process, as well as providing a link with percolation theory. The latter approach has the advantage of being more impressive to intimidate peers. Here, we will adopt the former.

Let

$$H = \{(\mathcal{R}^x)_{x \in V},\ (\mathcal{T}^e)_{e \in E}\}$$

be a family of independent Poisson point processes on $[0, \infty)$, all independent, as follows:

- each $\mathcal{R}^x$ with intensity 1 (*recovery times* at x, to be drawn as "×" marks over x);
- each $\mathcal{T}^{\{x,y\}}$ with intensity λ (*transmission times* from x to y, to be drawn as bridges between x and y.

Given H and an initial configuration, we can construct the process.

An *infection path* of H is a function $\gamma : I \to V$ (where I is a time interval) such that

- γ does not touch recovery marks, that is,
$$t \notin \mathcal{R}^{\gamma(t)} \text{ for all } t;$$
- γ can only jump by traversing bridges, that is,
$$\gamma(t) \neq \gamma(t-) \quad \Longrightarrow \quad t \in \mathcal{T}^{\{\gamma(t-), \gamma(t)\}}.$$

We adapt the following notation:

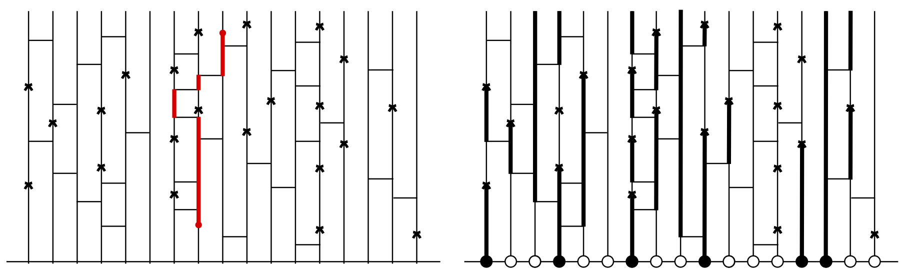
- given $x, y \in V$ and $s \leq t$, we write $(x, s) \rightsquigarrow (y, t)$ if there is an infection path $\gamma : [s, t] \to V$ with $\gamma(s) = x$, $\gamma(t) = y$;
- given $A \subseteq V$ and y, s, t as above, we write $A \times \{s\} \rightsquigarrow (y, t)$ if $(x, s) \rightsquigarrow (y, t)$ for some $x \in A$;
- similarly, with obvious meanings, we write $(x, s) \rightsquigarrow B \times \{t\}$ and $A \times \{s\} \rightsquigarrow B \times \{t\}$.

Then, given $A \subseteq V$, by setting

$$\xi_t^A(x) := \mathbb{1}\{A \times \{0\} \rightsquigarrow (x,t)\}, \quad x \in V,\ t \geq 0,$$

we obtain $(\xi_t^A)_{t\geq 0}$, the contact process on G with infection rate λ and $\xi_0 = A$.

In the figure below (where the graph is $\mathbb{Z}$), the left side shows an infection path, and the right side shows the evolution of the contact process (initially infected sites are represented by black balls, and infected areas are depicted with thicker black lines).



2.2 First properties

The graphical construction allows us to construct, in a single probability space, contact processes started from all possible initial configurations (*universal coupling*). It also makes many important properties easy to check. We now list some of them.

- *Additivity.* For any $A \subseteq V$, $A \neq \varnothing$, we have

$$\xi_t^A = \cup_{x\in A} \xi_t^{\{x\}}.$$

- *Attractivity.* For any $A, B \subseteq V$ with $A \subseteq B$, we have

$$\xi_t^A \leq \xi_t^B \quad \text{for all } t \geq 0.$$

  Hence, $(\xi_t^A)_{t\geq 0}$ is stochastically dominated by $(\xi_t^B)_{t\geq 0}$. It is also possible to check that if $\lambda < \lambda'$, then $(\xi_t^A)_{t\geq 0}$ with rate λ is stochastically dominated by $(\xi_t^A)_{t\geq 0}$ with rate λ'. To do so, we take a graphical construction $H = \{(\mathcal{R}^x), (\mathcal{T}^e)\}$, we sample an independent set of extra bridges $(\tilde{\mathcal{T}}^e)$ with rate $\lambda' - \lambda$, and evolve one of the process using only the briges in $(\mathcal{T}^e)$, whereas the larger process is also allowed to use the extra bridges.

- *Absorbing state.* If $\xi_t \neq \varnothing$, then $\xi_s = \varnothing$ for all $s \geq t$.

- *Duality.* For any $A, B \subseteq V$ and any $t \geq 0$, we have

$$\begin{aligned}\mathbb{P}(\xi_t^A \cap B \neq \varnothing) &= \mathbb{P}(A \times \{0\} \rightsquigarrow B \times \{t\}) \\ &= \mathbb{P}(B \times \{0\} \rightsquigarrow A \times \{t\}) = \mathbb{P}(\xi_t^B \cap A \neq \varnothing).\end{aligned}$$

  Applying this with $A = V$ and $B = \{x\}$ gives

$$\mathbb{P}(\xi_t^V(x) = 1) = \mathbb{P}(\xi_t^{\{x\}} \neq \varnothing).$$

- *Survival probability.* The probability $\mathbb{P}(\xi_t^A \neq \varnothing \ \forall t)$ is always equal to 1 when A is infinite, so it is interesting to consider it when A is finite. Provided that G is connected, either this probability is zero for every choice of (finite, non-empty) A, or it is non-zero for every choice of (finite, non-empty) A. The process is said to *die out* in the former case, and to *survive* in the latter case.

We also record the following as a lemma:

Lemma 2.2.1. *If all vertices of G have degree smaller than d and $\lambda < 1/d$, then for any finite $A \subseteq V$, $\mathbb{P}(\xi_t^A \neq \varnothing \ \forall t) = 0$.*

Proof. For $\xi \in \{0,1\}^V$, we write $|\xi| := \sum_x \xi(x)$, the number of infections in ξ. Assume that A is finite and non-empty. The process $(|\xi_t^A|)_{t \geq 0}$ is stochastically dominated by the continuous-time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{N}_0$ with $X_0 = |A|$ and jump rates

$$r(0,n) = 0 \ \forall n, \qquad \text{and for } n \geq 1,\ r(n,n-1) = n,\ r(n,n+1) = d\lambda n.$$

Let $t_0 = 0$ and, for $n \in \mathbb{N}_0$, $t_{n+1} := \inf\{t > t_n : X_t \neq X_{t_n}\}$. Then, $(X_{t_n})_{n\in\mathbb{N}_0}$ is a random walk on $\mathbb{N}_0$ that is absorbed at zero, and otherwise jumps to the left with probability $\frac{1}{1+d\lambda}$ and to the right with probability $\frac{d\lambda}{1+d\lambda}$. Since $1 > d\lambda$, this chain has a bias towards the right, so it is absorbed at zero almost surely. □

2.3 Phase transition on $\mathbb{Z}^d$

In the statement of the following theorem, let $\mathbb{P}_\lambda$ denote a probability under which the contact process on $\mathbb{Z}^d$ with rate λ is defined. Recall that $\lambda \mapsto \mathbb{P}_\lambda(\xi_t^{\{0\}} \neq \varnothing \ \forall t)$ is non-decreasing.

Theorem 2.1 (Harris [16])**.** *Letting*

$$\lambda_c = \lambda_c(\mathbb{Z}^d) := \sup\{\lambda : \ \mathbb{P}_\lambda(\xi_t^{\{0\}} \neq \varnothing \ \forall t) = 0\},$$

we have

$$\lambda_c \in (0, \infty).$$

Consequently,

$$\begin{aligned}\lambda < \lambda_c \quad &\Longrightarrow \quad \mathbb{P}(\xi_t^{\{0\}} \neq \varnothing \ \forall t) = 0 \quad \text{(subcritical regime);}\\ \lambda > \lambda_c \quad &\Longrightarrow \quad \mathbb{P}(\xi_t^{\{0\}} \neq \varnothing \ \forall t) > 0 \quad \text{(supercritical regime).}\end{aligned}$$

As mentioned in the Introduction, Bezuidenhout and Grimmett [4] proved that the process with $\lambda = \lambda_c$ (*critical regime*) dies out.

The figure below shows simulations of the contact process on $\mathbb{Z}$ in the subcritical and supercritical regimes.

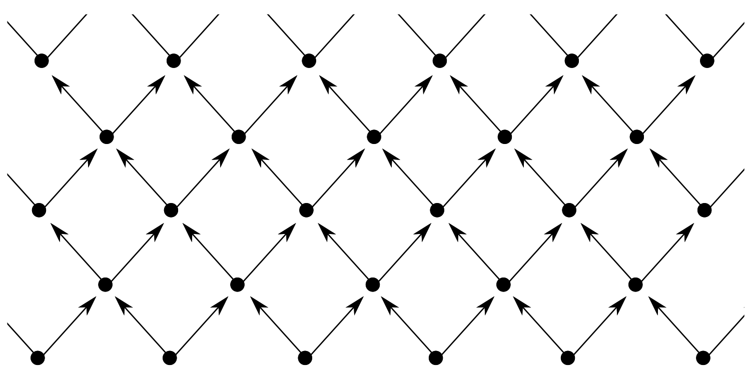
The fact that the process dies out when λ is small follows from Lemma 2.2.1. We'll see a sketch of proof of the fact that the process survives when λ is large. It suffices to prove this in dimension $d = 1$. Note that

$$\{\xi_t^{\{0\}} \neq \varnothing \ \forall t\} = \{(0,0) \rightsquigarrow \mathbb{Z} \times \{t\} \ \forall t\} =: \{(0,0) \rightsquigarrow \infty\}.$$

The proof is done by a comparison with *oriented percolation*.

2.4 Interlude: oriented percolation

There are many possible variants of oriented percolation (site/bond models, models on different lattices etc.); for the present purposes, let Φ be the set of bonds in the oriented graph:



For $\eta \in \{0,1\}^\Phi$ and $\vec{e} \in \Phi$, we say that $\vec{e}$ is *open* in η if $\eta(\vec{e}) = 1$, and otherwise that $\vec{e}$ is *closed* in η. We consider η a random element of $\{0,1\}^\Phi$, so that $\eta(\vec{e}) \sim \mathrm{Ber}(p)$ for all $\vec{e}$. We don't assume that $\{\eta(\vec{e}) : \vec{e} \in \Phi\}$ are independent, but rather, that they are *k-dependent* for some $k \in \mathbb{N}$, meaning that: for any m, $\eta(\vec{e}_1), \ldots, \eta(\vec{e}_m)$ are independent, provided that $\vec{e}_1, \ldots, \vec{e}_m$

are pairwise at distance at least k from each other (we say that the distance between $\vec{e}_i$ and $\vec{e}_j$ is the minimum of the Euclidean distance between x and y, with x being one of the two vertices of $\vec{e}_i$ and y one of the two vertices of $\vec{e}_j$).

Theorem 2.2. *For any $k > 0$ there exists $\bar{p}(k) \in (0,1)$ such that if $p > \bar{p}(k)$, then in any k-dependent oriented percolation model with the properties described above, with density parameter p, we have*

$$\mathbb{P}(\exists \textit{ infinite open path from the origin}) > 0.$$

Results of these kind are proved using *contour arguments*, which are classical in percolation theory. See Durrett [12].

2.5 Supercritical regime on $\mathbb{Z}^d$

We now return to proving that the contact process on $\mathbb{Z}^d$ can survive if the infection rate is sufficiently high. Fix $\delta > 0$. Let H be a graphical construction for the contact process on $\mathbb{Z}$. Define $\eta_H \in \{0,1\}^\Phi$ by setting, for each $\vec{e} = \langle (x,n),(y,n+1)\rangle \in \Phi$,

$$\eta_H(\vec{e}) = \mathbb{1}\left\{\begin{array}{l}\text{in the time interval } [\delta n, \delta(n+1)], \text{ there is no recovery} \\ \text{at } x \text{ or } y, \text{ and at least one transmission in } \{x,y\}\end{array}\right\}.$$

Then, $\{\eta_H(x,n) : x \in Z,\ n \in \mathbb{N}_0\}$ is a 2-dependent oriented percolation configuration. Moreover,

$$\{\exists \text{ infinite open path from the origin in } \eta_H\} \subseteq \{\xi_t^{\{0\}} \neq \varnothing \ \forall n\}.$$

Finally, η_H has density parameter

$$p = (e^{-\delta})^2 \cdot (1 - e^{-\delta\lambda}).$$

This can be made as close to 1 as desired by first taking δ small and then taking λ large. The proof is now complete.

A lot more is known about the supercritical contact process on $\mathbb{Z}d$. Here we only mention one highlight:

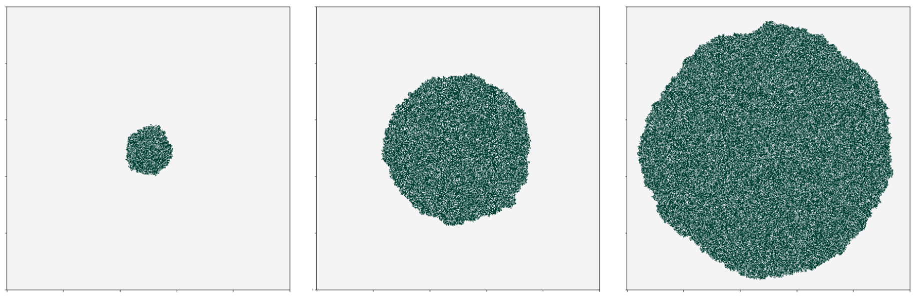
Theorem 2.3 (Shape theorem)**.** *Let $(\xi_t^{\{0\}})_{t\geq 0}$ be the contact process on $\mathbb{Z}^d$ with $\lambda > \lambda_c$. Define*

$$K_t := \bigcup_{s \leq t} \bigcup_{x \in \xi_s^{\{0\}}} (x + [-\tfrac{1}{2}, \tfrac{1}{2}]^d).$$

Then, there exists a (deterministic) compact and convex set $K \subset \mathbb{R}^d$ such that, for any $\epsilon > 0$,

$$\mathbb{P}\left((1-\epsilon)t \cdot K \subseteq K_t \subseteq (1+\epsilon)t \cdot K \mid \xi_t^{\{0\}} \neq \varnothing\right) \xrightarrow{t\to\infty} 1.$$

Below is a simulation of the contact process on $\mathbb{Z}^2$, illustrating the Shape theorem.



Finite-volume phase transition

If $G = (V, E)$ is a finite graph, then the contact process almost surely reaches the empty configuration. Indeed, the event E_n that in the time interval $[n, n+1]$, there is no transmission and there is a recovery at every vertex has probability at least $e^{-\lambda|E|} \cdot (1 - e^{-|1|})^{|V|}$. Since these events are independent, the Borel-Cantelli Lemma implies that $\mathbb{P}(\cup_n E_n) = 1$. If E_n occurs, then $\xi_t = \varnothing$ for all $t \geq n+1$, so $\cup_n E_n \subseteq \{\exists t :\ \xi_t = \varnothing\}$.

We define the *extinction time of the contact process on* G as the random variable

$$\tau_G := \inf\{t : \xi_t^V = \varnothing\} \tag{2.5.1}$$

(note that in this definition, we start the process from the configuration where every vertex is infected, which is a worst-case scenario).

The following has been proved about the contact process on boxes of $\mathbb{Z}^d$. The reason for the numerous citations in the statement is that this theorem was proved in several stages.

Theorem 2.4. *[13, 8, 33, 14, 27, 26] Let B_n be the finite subgraph of $\mathbb{Z}^d$ induced by the vertex set $\{-n, \dots, n\}^d$, and let $|B_n| = (2n+1)^d$ be its number of vertices.*

- *If $\lambda < \lambda_c(\mathbb{Z}^d)$, then there exists $c = c(\lambda) \in (0, \infty)$ such that*

$$\frac{\mathbb{E}[\tau_{B_n}]}{\log(|B_n|)} \xrightarrow{n\to\infty} c.$$

- *If $\lambda > \lambda_c(\mathbb{Z}^d)$, then there exists $C = C(\lambda) \in (0, \infty)$ such that*

$$\frac{\log \mathbb{E}[\tau_{B_n}]}{|B_n|} \xrightarrow{n\to\infty} C.$$

This theorem can be seen as a *finite-volume phase transition*, since it is the finite-graph counterpart of the phase transition of the contact process on $\mathbb{Z}^d$.

The supercritical case is often referred to as the *metastable regime* (*metastability* is the phenomenon where a system persists for a long time in a situation that resembles, but is not, an equilibrium, until eventually passing to the real equilibrium). Metastability for the contact process is a large and active topic of research.

3 The contact process on dynamic percolation

3.1 Percolation and dynamic percolation

The following is a brief exposition of some definitions concerning Bernoulli bond percolation and dynamic percolation on a graph. See [5] for an exposition on percolation theory, and [35] for a survey on dynamic percolation.

Let us start with (static) percolation. We let $G = (V, E)$ be a connected graph and $p \in [0, 1]$. Under a probability measure $\mathbb{P}_p$, we take independent random variables $\{\zeta_e : e \in E\}$, all with the Bernoulli(p) distribution. We say that the edge e is *open* if $\zeta_e = 1$, and that it is *closed* otherwise. We define the percolation event as

$$\text{Perc} := \{\text{the subgraph of } G \text{ induced by open edges has an infinite component}\}.$$

We then let

$$p_c := \sup\{p : \mathbb{P}_p(\text{Perc}) = 0\}.$$

It is easy to see that $p_c = 1$ when $G = \mathbb{Z}$, and it is well-known that $p_c \in (0, 1)$ when $G = \mathbb{Z}^d$ with $d \geq 2$.

Turning to dynamic percolation, fix $p \in [0, 1]$ and $\mathsf{v} > 0$. Assume that for each edge e of the graph, we are given a continuous-time Markov chain $(\zeta_t(e))_{t\geq 0}$ on $\{0, 1\}$ which jumps

$$\begin{aligned} &0 \to 1 \quad \text{with rate } p\mathsf{v}, \\ &1 \to 0 \quad \text{with rate } (1-p)\mathsf{v}. \end{aligned} \tag{3.1.2}$$

The transition probabilities of this chain are given by

$$\begin{pmatrix} P_t(0,0) & P_t(0,1) \\ P_t(1,0) & P_t(1,1) \end{pmatrix} = \begin{pmatrix} 1-p+pe^{-\mathsf{v}t} & p-pe^{-\mathsf{v}t} \\ 1-p-(1-p)e^{-\mathsf{v}t} & p+(1-p)e^{-\mathsf{v}t} \end{pmatrix}. \tag{3.1.3}$$

For different edges, these chains are assumed to be independent. The initial state ζ_0 can be arbitrary (deterministic or random), but typically we take $\zeta_0 \sim \pi_p = \otimes_{e\in E}\text{Ber}(p)$. Regardless of this choice, for any $t \geq 0$, ζ_t is a configuration for bond percolation on G.

3.2 The contact process on dynamic percolation

In what follows, $\lambda_c(G)$ will keep denoting the critical infection rate of the contact process on the (static) graph G.

The contact process on dynamic percolation on G is a Markov process $(\xi_t, \zeta_t)_{t\geq 0}$ on $\{0,1\}^V \times \{0,1\}^E$, where (ζ_t) is a dynamic percolation on G which evolves autonomously, and (ξ_t) is a contact process on G, except that at any given time, attempts to transmit the infection through closed edges are suppressed. The parameters of this process are then $p \in [0,1]$, $\mathsf{v} > 0$ (for the dynamic percolation), and $\lambda > 0$ (for the contact process).

A graphical construction can be taken for this process with

$$H = \{(\mathcal{O}^e)_{e\in E}, (\mathcal{C}^e)_{e\in E}, (\mathcal{T}^e)_{e\in V}, (\mathcal{R}^x)_{x\in V}\},$$

where:

- $\mathcal{O}^e$ has rate $p\mathsf{v}$ and contains the times at which e opens;
- $\mathcal{C}^e$ has rate $(1-p)\mathsf{v}$ and contains the times at which e closes;
- $\mathcal{T}^e$ has rate λ and contains the times when a transmission *is attempted* along e;
- $\mathcal{R}^x$ has rate 1 and contains the recovery times of x.

Let $\mathcal{U}^e := \mathcal{O}^e \cup \mathcal{C}^e$, the set of update times of edge e. Also define $\overline{\mathcal{T}}^e := \{t \in \mathcal{T}^e : \zeta_t(e) = 1\}$. The contact process is constructed as before using the recovery marks in $(\mathcal{R}^x)$ and the transmission marks in $(\overline{\mathcal{T}}^e)$; the transmission marks in $(\mathcal{T}^e \backslash \overline{\mathcal{T}}^e)$ are ignored.

3.3 Critical infection rate

Let $0 \in V$ be a distinguished vertex. We define

$$\lambda_c^{\mathrm{dyn}}(\mathsf{v}, p) := \inf\{\lambda > 0 : \ \mathbb{P}_{\mathsf{v},p,\lambda}(\xi_t \neq \varnothing \ \forall t \mid \xi_0 = \mathbb{1}_{\{o\}}, \ \zeta_0 \sim \pi_p) > 0\}.$$

The above definition involves a specific choice of initial state, and we want to argue that it doesn't matter.

Fix v, p, and let

$$f_\lambda(\xi_0, \zeta_0) := \mathbb{P}_{sfv,p,\lambda}(\xi_t \neq \varnothing \ \forall t \mid \xi_0, \zeta_0)$$

Note that

$$\lambda_c^{\mathrm{dyn}}(sfv, p) = \inf\{\lambda > 0 : \int f_\lambda(\mathbb{1}_{\{o\}}, \zeta) \ \pi_p(\mathrm{d}\zeta) > 0\}.$$

Theorem 3.1 (Seiler and Sturm, [34]). *Assume that G has polynomial growth (that is, there exist $C > 0$ and $k > 0$ such that for any radius r, the cardinality of the graph-distance ball of radius r around 0 is smaller than Cr^k). Fix $p \in (0,1]$ and $\mathsf{v} > 0$. Then,*

$$\textit{either} \quad f_\lambda(\xi_0, \zeta_0) = 0 \quad \textit{for all } \zeta_0 \textit{ and } \xi_0 \textit{ with } 0 < |\xi_0| < \infty$$

$$\textit{or} \quad f_\lambda(\xi_0, \zeta_0) > 0 \quad \textit{for all } \zeta_0 \textit{ and } \xi_0 \textit{ with } 0 < |\xi_0| < \infty.$$

Proof. We start with the **Claim:**

$$\forall \epsilon > 0 \, \exists R \in \mathbb{N}: \text{ if } \zeta_0, \zeta_0' \text{ agree inside } B_0(R), \qquad \text{then } |f_\lambda(\mathbb{1}_{\{o\}}, \zeta_0) - f_\lambda(\mathbb{1}_{\{o\}}, \zeta_0')| < \epsilon. \tag{3.3.4}$$

Let us prove this. Let $\epsilon > 0$. Fix R large, to be chosen later, and ζ_0, ζ_0' agreeing inside $B_0(R)$. Take a graphical construction H; use it to define $(\xi_t, \zeta_t)_{t \geq 0}$ and $(\xi_t', \zeta_t')_{t \geq 0}$, where $\xi_0 = \xi_0' = \mathbb{1}_{\{o\}}$. Additionally, use H to define $(\bar{\xi}_t)_{t \geq 0}$, the contact process started from $\mathbb{1}_{\{o\}}$ which can use all transmission marks, regardless of whether edges are open or closed. Note that for any t,

$$\xi_t \leq \bar{\xi}_t \quad \text{and} \quad \xi_t' \leq \bar{\xi}_t.$$

Now fix $\alpha > 0$ and define the good event

$$A_\alpha := \left\{ \begin{array}{l} \bar{\xi}_t \subseteq B_0(\frac{R}{2} + \alpha t) \text{ for any } t \text{ and} \\ \zeta_t(e) = \zeta_t'(e) \text{ for any } (e,t) \text{ with } e \subset B_0(\frac{R}{2} + \alpha t) \end{array} \right\}.$$

By first choosing $\alpha > 0$ large and then R large, we have $\mathbb{P}(A_\alpha) > 1 - \epsilon$, uniformly over all choices of ζ_0, ζ_0' that agree inside $B_0(R)$. If A_α occurs, then $\xi_t = \xi_t'$ for all t. Hence,

$$|f_\lambda(\mathbb{1}_{\{o\}}, \zeta_0) - f_\lambda(\mathbb{1}_{\{o\}}, \zeta_0')| \leq \mathbb{P}(\exists t: \; \xi_t \neq \xi_t') \leq \mathbb{P}(A_\alpha^c) < \epsilon.$$

This proves the Claim.

Now, assume that $f_\lambda(\mathbb{1}_{\{o\}}, \zeta_0) > 0$ for some ζ_0. Then, $f_\lambda(\mathbb{1}_{\{0\}}, \mathbb{1}) > 0$. This implies that there is $R > 0$ such that $f_\lambda(\mathbb{1}_{\{0\}}, \mathbb{1}_{B_0(R)}) > 0$. Now take some $\zeta_0' \in \{0,1\}^E$. For the process started from $(\mathbb{1}_{\{0\}}, \zeta_0')$, with positive probability, after one time unit, (1) the contact process doesn't change, and (2) all edges inside $B_0(R)$ become open. Conditionally on this, the process then survives with positive probability.

We have thus proved that

$$f_\lambda(\mathbb{1}_{\{0\}}, \zeta_0) > 0 \text{ for some } \zeta_0 \quad \Leftrightarrow \quad f_\lambda(\mathbb{1}_{\{0\}}, \zeta_0) > 0 \text{ for all } \zeta_0.$$

It is easy to extend this to

$$f_\lambda(\mathbb{1}_{\{x\}}, \zeta_0) > 0 \text{ for some } x, \zeta_0 \quad \Leftrightarrow \quad f_\lambda(\mathbb{1}_{\{0\}}, \zeta_0) > 0 \text{ for all } x, \zeta_0,$$

and then to conclude by using:

$$f_\lambda(\xi_0, \mathbb{1}) \leq \sum_{x \in \xi_0} f_\lambda(\mathbb{1}_{\{x\}}, \mathbb{1}).$$

□

Going back to the definition of $\lambda_c^{\text{dyn}}(\mathsf{v}, p)$, we have the obvious monotonicity facts:

$$\lambda_c^{\text{dyn}}(\mathsf{v}, p) \geq \lambda_c$$

and

$$p \leq p' \quad \Longrightarrow \quad \lambda_c^{\text{dyn}}(\mathsf{v}, p) \geq \lambda_c^{\text{dyn}}(\mathsf{v}, p').$$

It is not clear whether monotonicity in v should hold.

3.4 Known results

From now on, assume that G is vertex transitive (that is, for any two vertices u and v of G, there exists a graph automorphism of G that maps u to v).

Theorem 3.2 (Linker and Remenik, [25]). *For any $p \in [0,1]$,*

$$\lim_{\mathsf{v}\to\infty} \lambda_c^{\mathrm{dyn}}(\mathsf{v},p) = \frac{\lambda_c}{p}.$$

This is natural: if v is large, then the process behaves like a contact process with infection rate λp.

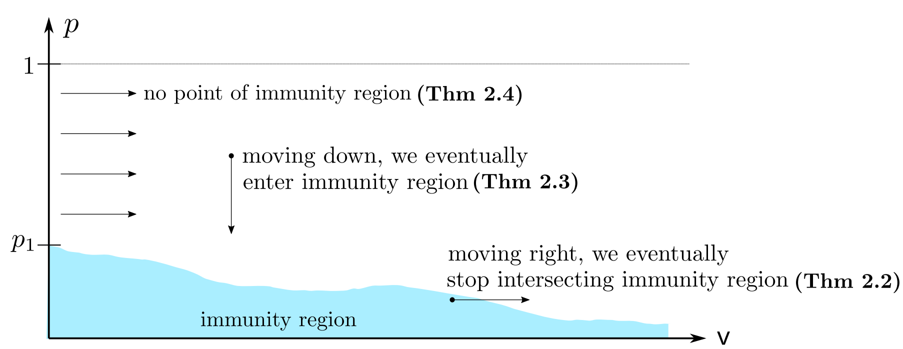
Theorem 3.3 (Linker and Remenik, [25]). *For all $\mathsf{v} > 0$, if p is small enough we have $\lambda_c^{\mathrm{dyn}}(\mathsf{v},p) = \infty$.*

When $\lambda_c^{\mathrm{dyn}}(\mathsf{v},p) = \infty$, we say that the dynamic network is immune to the infection. The *immunity region* is defined as

$$\{(\mathsf{v},p) : \lambda_c^{\mathrm{dyn}}(\mathsf{v},p) = \infty\}.$$

Theorem 3.4 (Linker and Remenik, [25]). *There exists $p_1 > 0$ such that if $p > p_1$, then $\lambda_c(p,\mathsf{v}) < \infty$ for any $\mathsf{v} > 0$.*

The graph below has a sketch of the set of parameters, including the immunity region and what the three theorems say about it. Note that nothing is said about the shape of the curve that delimits the immunity region from above; in particular, it is not known to be decreasing.



The following is the only theorem of [25] where the authors need to assume the graph to be $\mathbb{Z}$.

Theorem 3.5 (Linker and Remenik, [25]). *Assume that $G = \mathbb{Z}$. For any $p \in [0,1)$, we have*

$$\lim_{\mathsf{v}\to\infty} \lambda_c^{\mathrm{dyn}}(\mathsf{v},p) = \infty.$$

The authors ask about analogous results on $\mathbb{Z}^d$ with $d > 1$. Later, the following was proved.

Theorem 3.6 (Hilário, Ungaretti, V., Vares, [17]). *On $\mathbb{Z}^d$ with $d \geq 2$,*

(a) *if $p < p_c$, then $\lim_{\mathsf{v} \to 0} \lambda_c^{\mathrm{dyn}}(\mathsf{v}, p) = \infty$;*

(b) *if $p > p_c$, then $\mathsf{v} \mapsto \lambda_c^{\mathrm{dyn}}(\mathsf{v}, p)$ is bounded.*

This theorem doesn't say much about the immunity region. The following is of interest.

Question. Is it the case that for any $p < p_c$, we have $\lambda_c^{\mathrm{dyn}}(\mathsf{v}, p) = \infty$ for v small enough?

An affirmative answer would mean that, in the above figure, we have $p_c = p_1$.

We now turn to the proofs of the theorems. In some cases the proof will be given in detail, while in others we will only explain the main ideas.

3.5 Proof of Theorem 3.2, survival part

In this section, we will present part of the proof of Theorem 3.2, namely, we will show that for any $\lambda > 0$ and $p \in (0, 1]$ with $\lambda p > \lambda_c$, if v is large enough, then there is survival at (v, p, λ).

The proof is an immediate consequence of a stochastic domination result due to Erik Broman.

Theorem 3.7 (Broman, [6]). *Let $(B_t)_{t \geq 0}$ be the continuous-time Markov chain on $\{0, 1\}$ with rates as in (3.1.2), with $B_0 \sim \mathrm{Ber}(p)$. Let $\bar{X}$ be a Poisson point process on $[0, \infty)$ with intensity λ. Define $X := \{t \in \bar{X} : B_t = 1\}$. Then, $(X_t)_{t \geq 0}$ stochastically dominates a Poisson point process on $[0, \infty)$ with intensity*

$$\beta(\mathsf{v}, p, \lambda) := \frac{1}{2}\left(\mathsf{v} + \lambda - \sqrt{(\mathsf{v} + \lambda)^2 - 4p\mathsf{v}\lambda}\right).$$

In view of this, the contact process on dynamical percolation with parameters (v, p, λ) and $\zeta_0 \sim \pi_p$ stochastically dominates a classical contact process with rate $\beta(\mathsf{v}, p, \lambda)$.

We note the following:

$$\begin{aligned}
\beta(\mathsf{v}, p, \lambda) &= \frac{1}{2}\left(\sqrt{(\mathsf{v} + \lambda)^2} - \sqrt{(\mathsf{v} + \lambda)^2 - 4p\mathsf{v}\lambda}\right) \\
&= \frac{1}{2}\int_{(\mathsf{v}+\lambda)^2 - 4p\mathsf{v}\lambda}^{(\mathsf{v}+\lambda)^2} \frac{1}{2\sqrt{x}}\,\mathrm{d}x \\
&\in \left[\frac{1}{2} \cdot 4p\mathsf{v}\lambda \cdot \frac{1}{2(\mathsf{v} + \lambda)},\ \frac{1}{2} \cdot 4p\mathsf{v}\lambda \cdot \frac{1}{2\sqrt{(\mathsf{v} + \lambda)^2 - 4p\mathsf{v}\lambda}}\right];
\end{aligned}$$

as $\mathsf{v} \to \infty$, the above interval approaches $\{p\lambda\}$. Hence, given p, λ with $p\lambda > \lambda_c$, we have $\beta(\mathsf{v}, p, \lambda) > \lambda_c$ if v is large, so the process survives if v is large, as required.

3.6 Proof of Theorem 3.2, extinction part

We will now prove that if λ, p are such that $\lambda p < \lambda_c$, then if v is large enough, the process dies out at (v, p, λ).

By elementary monotonicity considerations, it is sufficient to prove extinction of the process assuming that $\zeta_0 \equiv 1$.

We take the graphical construction $H = \{(\mathcal{R}^x), (\mathcal{T}^e), (\mathcal{O}^e), (\mathcal{C}^e)\}$, with $\mathcal{U}^e = \mathcal{O}^e \cup \mathcal{C}^e$. Let us say that an edge e is *polluted* at time t if there is $s \in [0,t] \cap \mathcal{T}^e$ such that $\mathcal{U}^e \cap [s,t] = \varnothing$ (that is: the most recent ocurrence in that edge was a transmission).

We define the auxiliary process $(\xi'_t)_{t\geq 0}$ which evolves exactly like $(\xi_t)_{t\geq 0}$, except that it is not allowed to transmit the infection through polluted edges.

Claim 3.6.1. *(ξ'_t) dies out.*

Proof. When $\zeta_0 \sim \pi_p$, (ξ'_t) is stochastically dominated by a contact process with rate $p\mathsf{v} < \lambda_c$, so it dies out. We now need to show that it also dies out when $\zeta_0 \equiv 1$. So from now on, we assume that $\zeta_0 \equiv 1$. Fix $p' > p$ still satisfying $p'\lambda < \lambda_c$. Fix t_0 large enough that the law of ζ_{t_0} is stochastically dominated by $\pi_{p'}$ (this is possible, since the environment converges in distribution to $\pi_p \preceq \pi_{p'}$.

We define another auxiliary process $(\xi''_t)_{t\geq 0}$ as follows:

- from time 0 to time t_0, it ignores the environment, and evolves as a contact process;
- from t_0 onwards, it behaves as (ξ'_t), with all edges declared unpolluted at time t_0.

Since the law of ζ_{t_0} is stochastically dominated by $\pi_{p'}$, (ξ''_t) dies out (by the same argument that was given when the environment was started from π_p). It is easy to see that (ξ''_t) stochastically dominates (ξ'_t), so the latter also dies out. □

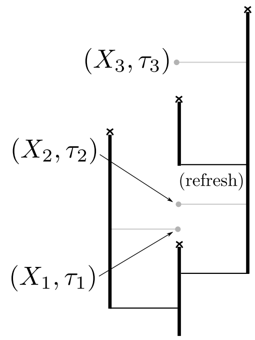
Now define

$$\tau_1 := \inf \left\{ \begin{array}{r} t: \text{ at time } t,\, \xi' \text{ attempts to transmit the infection to} \\ \text{a healthy site through a polluted edge} \end{array} \right\}$$

and, for $n \in \mathbb{N}_0$,

$$\tau_{n+1} := \inf \left\{ \begin{array}{r} t > \tau_n: \text{ at time } t,\, \xi' \text{ attempts to transmit the infection to} \\ \text{a healthy site through a polluted edge} \end{array} \right\},$$

with $\tau_{n+1} = \infty$ if $\tau_n = \infty$. Also let X_n be the position of the "target" of the attempted transmission at time t.



We now compute

$$\begin{aligned} S := \mathbb{P}(\xi \text{ survives}) &= \mathbb{P}(\xi \text{ survives}, \xi' \text{ dies}) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\tau_n < \infty,\ (X_n, \tau_n) \rightsquigarrow \infty) \\ &\le \sum_{n=1}^{\infty} \mathbb{P}(\tau_n < \infty) \cdot S = S \cdot \mathbb{E}[\#\{n : \tau_n < \infty\}]. \end{aligned}$$

It can be proved that

$$\mathsf{v} \text{ large enough} \quad \Longrightarrow \quad \mathbb{E}[\#\{n : \tau_n < \infty\}] < 1. \tag{3.6.5}$$

With this at hand, the above shows that when v is large, S is smaller than or equal to S times a constant smaller than 1, so S must be 0.

The proof of (3.6.5) is a little technical, so we won't cover it here. It requires us to prove that the expectation is finite in the first place. This is where we need the assumption that the graph is vertex transitive. A theorem of Aizenman and Jung [1] says that for the subcritical contact process on a (static) vertex transitive graph, the expected number of infections ever created is finite.

3.7 Proof of Theorem 3.3

Lemma 3.7.1. *Let* $\mathsf{v} > 0$ *and* $\epsilon > 0$. *If* T *is large enough (depending on* v *and* ϵ*) and* p *is small enough (depending on* v, ϵ, T*), the following holds. Letting* $(X_t)_{t \ge 0}$ *be the chain as in* (3.1.2) *with* $X_0 \sim \mathrm{Ber}(p)$ *and*

$$V_n := \mathbb{1}\{\exists t \in [nT, (n+1)T] : \ X_t = 1\}, \quad n \in \mathbb{N}_0,$$

we have

$$\mathbb{P}(V_0 = 1) < \epsilon, \quad \mathbb{P}(V_{n+1} = 1 \mid V_0, \dots, V_n) < \epsilon \text{ almost surely } \forall n.$$

Proof. It suffices to treat $\mathsf{v} = 1$ (a time rescaling takes care of other values). Fix $\epsilon > 0$. We have

$$\mathbb{P}(V_0 = 1) = p + (1-p)e^{-pT} \xrightarrow{p\to 0} 1.$$

For $n \geq 0$, we want to prove that $\mathbb{P}(V_{n+1} = 0 \mid V_0, \ldots, V_n) > 1 - \epsilon$. For this, it is sufficient to prove that $\mathbb{P}(V_{n+1} = 0 \mid V_0, \ldots, V_n, X_{nT}) > 1-\epsilon$. By the Markov property, this will follow from proving that

$$\mathbb{P}(V_{n+1} = 0 \mid X_{nT}, V_n) > 1 - \epsilon.$$

In order to do so, we consider all possible values of the pair (X_{nT}, V_n) that appears in the conditioning.

- $(X_{nT}, V_n) = (0, 0)$: this one is immediate:

$$\mathbb{P}(V_{n+1} = 0 \mid X_{nT} = 0, V_n = 0) = e^{-pT}.$$

- $(X_{nT}, V_n) = (1, 0)$ can be disregarded, as it is impossible.

- $(X_{nT}, V_n) = (0, 0)$: we have

$$\begin{aligned}\mathbb{P}(V_{n+1} = 0 \mid X_{nT} = 1, V_n = 1) &= \mathbb{P}(V_{n+1} = 0 \mid X_{nT} = 1)\\ &= (1 - e^{-T}) \cdot (1-p) \cdot e^{-pT};\end{aligned}$$

  the term $1 - e^{-T}$ is the probability of at least one arrival in $[nT, (n+1)T]$; the term $1 - p$ is for the last arrival to be 0, and e^{-pT} is for the process to stay 0 in $[(n+1)T, (n+2)T]$.

- $(X_{nT}, V_n) = (0, 0)$. This case is somewhat trickier, and we leave it as an exercise to show that

$$\mathbb{P}(V_{n+1} = 0 \mid X_{nT} = 0, V_n = 1) = \left(1 - p\frac{1 - e^{-T}}{1 - e^{-pT}}\right) e^{-pT}.$$

  The transition probabilities in (3.1.3) can help.

Putting all the bounds we have obtained together, we have that

$$\begin{aligned}\mathbb{P}(V_{n+1} = 0 \mid X_{nT}, V_n) &\geq (1-p)(1-e^{-T}) \left(1 - p\frac{1 - e^{-T}}{1 - e^{-pT}}\right) e^{-pT}\\ &\geq \underbrace{(1-p)}_{\xrightarrow{p\to 0} 1} \; (1 - e^{-T}) \; \underbrace{\left(1 - \frac{p}{1 - e^{-pT}}\right)}_{\xrightarrow{p\to 0} 1 - \frac{1}{T}} \; \underbrace{e^{-pT}}_{\xrightarrow{p\to 0} 1}.\end{aligned}$$

□

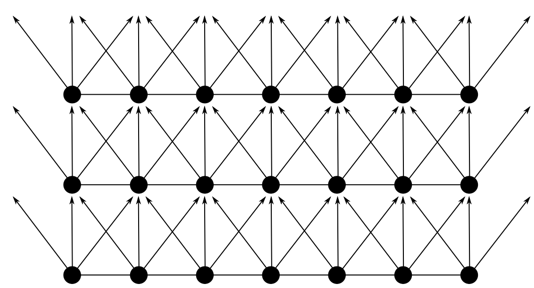
Let $G = (V, E)$ be a connected graph with degrees bounded by d, and let v be fixed. We will show that if p is small enough, then even the "process with $\lambda = \infty$" dies out. This is the process in which only vertices that are isolated (their connected component contains no one else) can recover, and the infection instantly occupies a whole connected component whenever it can travel there. We assume that $\xi_0 = \mathbb{1}_{\{u_0\}}$, where u_0 is an arbitrary vertex that we fix.

We take a graphical construction $H = \{(\mathcal{R}^x), (\mathcal{T}^e), (\mathcal{O}^e), (\mathcal{C}^e)\}$ as before. It induces a bound percolation configuration on the graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} := V \times \mathbb{N}_0$ and

$$\begin{aligned}\hat{E} :=& \{\{(u,n),(v,n)\} : \ \{u,v\} \in E, \ n \in \mathbb{N}_0\} \\ &\cup \{\langle (u,v),(v,n+1)\rangle : \ \{u,v\} \in E, \ n \in \mathbb{N}_0\} \\ &\cup \{\langle (u,v),(u,n+1)\rangle : \ u \in V, \ n \in \mathbb{N}_0\}\end{aligned}$$

(the first set in the union contains unoriented edges, and the second and third contain oriented edges).

As an example, if $G = \mathbb{Z}$, then $\hat{G}$ is the graph depicted below.



Let us describe how H produces a bond percolation configuration on $\hat{G}$. There are two rules for opening edges of $\hat{G}$:

Rule 1: If

$$\mathcal{R}^u \cap [nT, (n+1)T] = \varnothing,$$

then open the edge

$$\langle (u,n), (u, n+1)\rangle.$$

Rule 2: For $e = \{u, v\} \in E$, if

$$\zeta_t(e) = 1 \text{ for some } t \in [nT, (n+1)T],$$

then open the edges

$$\{(u,n),(v,n)\}, \qquad \langle (u,n),(v,n+1)\rangle, \qquad \langle (v,n),(u,n+1)\rangle.$$

All edges of $\hat{G}$ that are unaffected by these two rules are left closed.

We leave it as an exercise to check that if there is no infinite open path started at $(u_0, 0)$ in this percolation model, then the contact process started from $\mathbb{1}_{\{u_0\}}$ and constructed from H dies out.

Let us say that edges of $\hat{E}$ of the form $\{(u,n),(v,n)\}$, $\langle (u,n),(u,n+1)\rangle$ or $\langle (u,n),(v,n+1)\rangle$ are *at height n*. Using Lemma 3.7.1, given $\epsilon > 0$, we can choose first $T > 0$ and then $p > 0$ such that, for any edge $\hat{e} \in \hat{E}$ at height n,

$$\mathbb{P}(\hat{e} \text{ is open} \mid \text{status of all edges at height} < n) < \epsilon. \tag{3.7.6}$$

We now prove that if ϵ is small enough, then $(u_0, 0)$ is almost surely not connected to infinity in this auxiliary percolation model. The proof will involve a path-counting argument. Let

$$\gamma = ((u_0,t_0),(u_1,t_1),\ldots,(u_n,t_n))$$

be a path in $\hat{G}$, where $t_0 = 0$ and γ may traverse unoriented edges in any direction, and may traverse oriented edges only in the correct direction. Then, (3.7.6) implies that

$$\mathbb{P}(\gamma \text{ is open}) < \epsilon^n.$$

We now complete the proof by bounding

$$\begin{aligned}\mathbb{P}(\exists \text{open path of length } n \text{ started from } (u_0,0)) &\leq \sum_n \sum_{\gamma:|\gamma|=n} \mathbb{P}(\gamma \text{ is open}) \\ &\leq \sum_n \epsilon^n \cdot (2d+1)^n \xrightarrow{\epsilon\to 0} 0.\end{aligned}$$

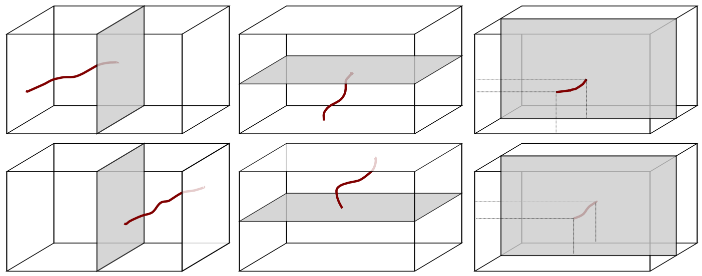
This concludes the proof of Theorem 3.3. The proof of Theorem 3.4 is done by a comparison with oriented percolation in the same spirit as the above.

3.8 Sketch of proof of Theorem 3.6(a)

We let $d \geq 2$ and fix $\lambda > 0$ and $p < p_c$. We want to show that if v is small enough, then the infection dies out almost surely.

This proof involves a renormalization scheme based on the graphical construction. It is quite technical, so we only see a very simplified picture of what is involved.

The construction will involve *space-time boxes*, which are translations of the box $[0,\ell]^d \times [0,h] \subseteq \mathbb{Z}^d \times [0,\infty)$. A *half-crossing* of such a box is an infection path, in the graphical construction, that traverses half of the box in one of the $d+1$ possible directions, as in the following picture, in $\mathbb{Z}^2 \times [0,\infty)$:



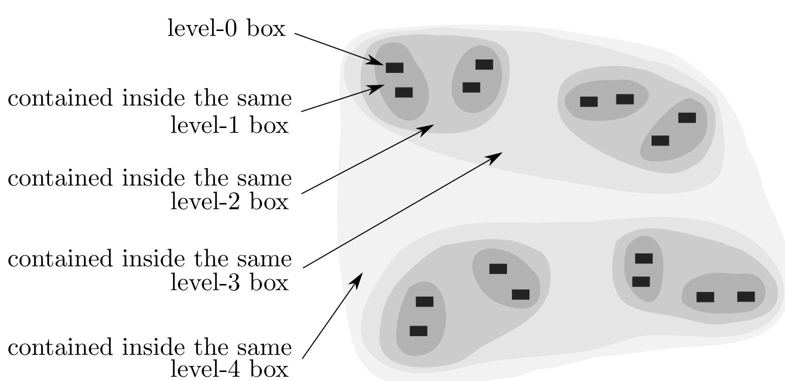
There will be *geometrically growing scales*. We choose *base scales* ℓ_0 (for space) and h_0 (for time) and set

$$\ell_k := \alpha^k \cdot \ell_0, \qquad h_k := \beta^k \cdot h_0$$

for $k \in \mathbb{N}$, where $\alpha, \beta > 1$ are *spacial and temporal growth factors.* A translation of $[0, \ell_k]^d \times [0, h_k]$ is called a *level-k box.*

The goal of the proof is to show that, if v is fixed at a small enough value, then the probability that a level-k box is half-crossed tends to 0 as $k \to \infty$. It is then straightforward to prove that the infection dies out.

The argument to prove this is recursive, as follows. We argue that if a level-k box is half-crossed, then two level-$(k-1)$ boxes inside it are also half-crossed, and moreover, these smaller boxes are far from each other (in the Euclidean norm of $\mathbb{R}^{d+1}$). This last condition guarantees that the dependence between the environments inside the two sub-boxes is weak. We then repeat this idea: by being half-crossed, each of the two level-$(k-1)$ sub-boxes must have further two level-$(k-2)$ sub-boxes that are half-crossed and far from each other, etc. Upon continuing this until level 0, we end up with a collection $\mathcal{X}$ of level-0 sub-boxes that are spread-out in space in a fractal manner, as in the picture:



We implement this reasoning in the form of a large union bound, like so: given a level-k box Q_k,

$$\mathbb{P}(Q_k \text{ is half-crossed}) \leq \sum_{\mathcal{X}} \mathbb{P}(\text{all boxes of } \mathcal{X} \text{ are half-crossed})$$
$$\leq \#\text{ways to choose } \mathcal{X} \cdot \sup_{\mathcal{X}} \mathbb{P}(\text{all boxes of } \mathcal{X} \text{ are half-crossed}).$$

At this point, there is a competition between the combinatorial term (number of ways to choose $\mathcal{X}$, which grows to infinity rapidly with k) and the probabilistic term (probability that in a given choice for $\mathcal{X}$, all level-0 boxes are half-crossed, which goes to zero rapidly with k when v is small).

The complicating factor is that there is dependence between the environments of the boxes of $\mathcal{X}$, when they are on top of each other. Although we have several constants to play with, including v, which we can take to zero, we have to be careful not to make the dependence too bad (lowering v does worsen it!).

To be precise, the constants we can choose are:

- α and β (growth coefficients for space and time; they also regulate the distance between the blobs in the fractal picture above);
- ℓ_0 and h_0 (size of level-0 box);
- v.

The order in which we choose them is as follows:

1. $\alpha = 4$ (it turns out that α is unimportant, so any other larger natural number would equally work);
2. ℓ_0 large, so that the clusters at the floor of a level-0 box have fewer than $C\log(\ell_0)$ vertices with high probability;
3. h_0 large, so that, if there were no edge updates in the space-time box, then the infection in all the clusters inside it would die;
4. β large, so that boxes that are vertically aligned are very far from each other, and v small, so that the density of edges that update inside a box is tiny.

The inclusion of β and v in the same item is on purpose: they are actually chosen together. The reason is that, as explained above, if we choose v too small compared to β, and if two boxes in $\mathcal{X}$ are on top of each other, then there is not enough space in between them for the environment to refresh.

4 The contact process on dynamic regular graphs

4.1 Preliminaries: the contact process on the infinite regular tree

Let G be a connected graph, and for $\lambda > 0$, let $\mathbb{P}_\lambda$ be a probability measure under which we have defined a graphical construction for the contact process with rate λ on G. Consistently with what we have done in Section 2, we can define

$$\lambda_c(G) := \sup\{\lambda : \mathbb{P}_\lambda(\xi_t^{\{o\}} \neq \varnothing \; \forall t) = 0\},$$

where o is an arbitrary vertex (as we have discussed before, when G is connected, this definition does not depend on the choice of o). This critical value can be called the *threshold for survival.* We can also define the *threshold for strong survival*,

$$\lambda_c^{\mathrm{str}}(G) := \inf\{\lambda : \mathbb{P}_\lambda\left(\liminf_{t\to\infty} \xi_t^{\{o\}}(o) = 1\right) > 0\}.$$

When λ is above this threshold, the infection has positive probability not only of surviving, but also of coming back to any fixed site infinitely many times. Obviously, $\lambda_c^{\mathrm{str}} \geq \lambda_c$.

It turns out that

$$\lambda_c(\mathbb{Z}^d) = \lambda_c^{\mathrm{str}}(\mathbb{Z}^d);$$

in $d = 1$, this follows from a comparison with oriented percolation, and in higher dimension, it can be obtained as a consequence of the Bezuidenhout–Grimmett renormalization.

A major contribution from Pemantle [29] was to show that for the infinite d-regular tree $\mathbb{T}^d$, with $d \geq 3$,

$$\lambda_c(\mathbb{T}^d) < \lambda_c^{\mathrm{str}}(\mathbb{T}^d),$$

so there is an intermediate parameter regime where the infection can survive, but any given vertex eventually becomes free from it for good. We mention this only for general knowledge: for the purposes of the rest of this course, the relevant critical parameter will continue being $\lambda_c^{(}\mathbb{T}^d)$.

4.2 Preliminaries: the contact process on the random d-regular graph

The random d-regular graph, to be defined below, is a class of finite random graphs that converge locally (in the sense of the Benjamini–Schramm local graph limit, [2]) to the infinite d-regular tree $\mathbb{T}^d$. For this reason, it is natural to expect that the contact process on these graphs exhibits a finite-volume phase transition, as in Theorem 2.4 for boxes of $\mathbb{Z}^d$. Since there are two critical values for $\mathbb{T}^d$, one may wonder which of them is relevant for this finite-volume phase transition, and it turns out that it is the smaller one, $\lambda_c(\mathbb{T}^d)$.

Let us define the random graph. Fix $d \in \mathbb{N}$, $d \geq 3$; this will be the degree of all vertices in the graph. Let $n \in \mathbb{N}$ be the number of vertices; we assume that dn is even. We let $V_n := \{1, \ldots, n\}$. We endow each vertex with d half-edges, which formally means that we take the *set of half-edges* $\mathcal{H} := \{(v, h) : v \in V_n,\ h \in \{1, \ldots, d\}\}$. Next, we take a perfect matching of $\mathcal{H}$ (that is, uniformly at random, we choose a partition of $\mathcal{H}$ into sets of two elements). For each two half-edges that are paired in this way, we form an edge (that is, if (u, a) and (v, b) are paired, we form an edge between u and v). This produces the random edge set E_n. Note that technically, this produces a *multi-graph*, because loops and parallel edges are possible. That is fine by us.[1] We will however largely ignore this. For the purposes of the contact process, loops are irrelevant, and whenever there are parallel edges, we see each edge as a separate venue for transmission (so that, if there are k edges between u and v, then u infects v with rate $k\lambda$).

The following has been proved (the two references proved this theorem at the same time independently). Recall that τ_G denotes the extinction time of the contact process on G, as in (2.5.1).

Theorem 4.1. *[22, 28]*

- *If $\lambda < \lambda_c(\mathbb{T}^d)$, then there exists $C > 0$ such that*

$$\mathbb{E}[\tau_{G_n}] \leq C \log(n) \quad \textit{for all } n.$$

- *If $\lambda > \lambda_c(\mathbb{T}^d)$, then there exists $c > 0$ such that*

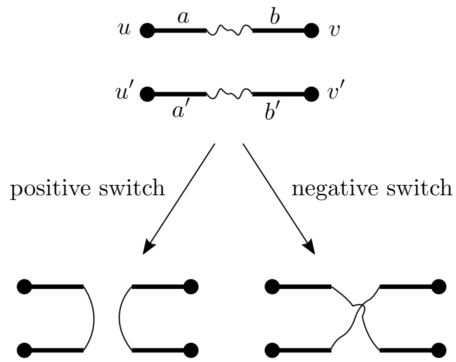
$$\mathbb{E}[\tau_{G_n}] \geq e^{cn} \quad \textit{for all } n.$$

4.3 Switching dynamics

An *edge-switching dynamics* on G_n was introduced by Cooper, Dyer and Greenhill [10] to study mixing time questions. This dynamics produces a continuous-time Markov chain $(G_n(t))_{t\geq 0}$ on the set of d-regular graphs with vertex set $V_n = \{1, \dots, n\}$ (we keep seeing a set of edges as a pairings of half-edges).

Let us introduce some terminology. Fix $n \in \mathbb{N}$ and let G be a random d-regular graph with n vertices. Let $e = \{(u, a), (v, b)\}$ and $e' = \{(u', a'), (v', b')\}$ be two edges of E. Assume that $(u, a) < (v, b)$ and $(u', a') < (v', b')$ in the lexicographic order on $\{1, \dots, n\} \times \{1, \dots, n\}$. The *switch with mark* $\mathsf{m} = (e, e', +)$ is the transformation Γ^{m} that maps the graph G into the graph $\Gamma^{\mathsf{m}}(G)$, which is equal to G except that the edges e and e' are removed, and replaced by the two new edges $\{(u, a), (u', a')\}$ and $\{(v, b), (v', v')\}$. This is called a *positive switch* (because it respects the lexicographic order). The *negative switch with mark* $\mathsf{n} = (e, e', -)$ maps G into $\Gamma^{\mathsf{n}}(G)$, where e, e' are replaced by $\{(u, a), (v', b')\}$ and $\{(v, b), (u', a')\}$.

[1] One minor nuissance is that we cannot represent E_n as a set of elements of the form $\{u, v\}$, but rather, we should keep seeing an edge as an object of the form $\{(u, a), (v, b)\}$, where $(u, a), (v, b) \in \mathcal{H}$.



We define $(G_n(t))_{t\geq 0}$ as the continuous-time Markov chain that at any given time, performs any of the possible switches with rate $\frac{\mathsf{v}}{nd}$. Here, $\mathsf{v} > 0$ is a parameter of the model. The reason we take the jump rate as $\frac{\mathsf{v}}{nd}$ is that we want v to be (at least approximately, as $n \to \infty$) the total rate at which a fixed edge is involved in a switch. To see why this works out, fix an edge e; the number of other edges is $\frac{nd}{2} - 1$, so the total number of possible switches involving e is $2 \cdot (\frac{nd}{2} - 1)$. Hence, the total rate of a switch involving e becomes $\frac{\mathsf{v}}{nd} \cdot (nd - 2) = \mathsf{v} - \frac{2\mathsf{v}}{nd}$.

A key feature of the edge-switching dynamics is that it is reversible with respect to the uniform measure on the set of all d-regular graphs on n vertices (the detailed balance equation is readily verified).

4.4 The contact process on dynamic regular graphs

We now fix $\lambda > 0$ and take the joint process $(G_n(t), \xi_t)_{t\geq 0}$, where (ξ_t) is the contact process evolving on the dynamic graph. We start the contact process start from all infected, and let τ_n denote the extinction time of the infection, as before.

Theorem 4.2 (Baptista da Silva, Oliveira, V. [11], Schapira, V.[32])**.** *For any* $\mathsf{v} > 0$*, there exists* $\bar{\lambda}(\mathsf{v}) \in (0, \lambda_c(\mathbb{T}^d))$ *such that:*

- *if* $\lambda < \bar{\lambda}(\mathsf{v})$*, then there exists* $C > 0$ *such that*

$$\mathbb{E}_{\mathsf{v},\lambda}[\tau_n] < C \log(n) \quad \textit{for all } n;$$

- *if* $\lambda > \bar{\lambda}(\mathsf{v})$*, then there exists* $c > 0$ *such that*

$$\mathbb{E}_{\mathsf{v},\lambda}[\tau_n] > e^{cn} \quad \textit{for all } n.$$

One key aspect of the above statement is that $\bar{\lambda}(\mathsf{v}) < \lambda_c(\mathbb{T}^d)$. In light of Theorem 4.1, this says that if $\lambda \in (\bar{\lambda}(\mathsf{v}), \lambda_c(\mathbb{T}^d))$, then the contact process dies in logarithmic time in the static graph, and survives for exponentially long in the dynamic graph. This is a confirmation that the graph dynamics helps the infection in this context.

In this direction, the following stronger result was also proved.

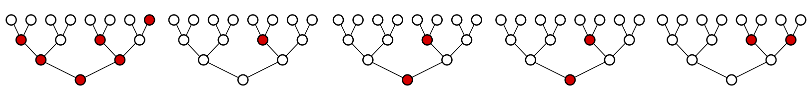
Theorem 4.3. *The function* $\mathsf{v} \mapsto \bar{\lambda}(\mathsf{v})$ *is strictly decreasing.*

We will not see the proof of Theorem 4.2. We will see the proof of Theorem 4.3 only for the *limiting dynamics* (as $n \to \infty$) that arises as the contact process (started from a single infection) evolves on this random graph model. Restricting ourselves to this simplified setting will allow us to focus on the monotonicity arguments and to avoid the random graph technicalities.

4.5 The herds process

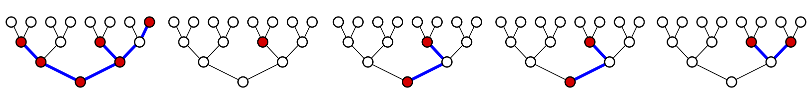
The herds process is easy to understand informally, but somewhat clumsy to define mathematically. Here we provide only an informal description of the dynamics, but we refer the reader to [32] for a full definition.

At any point in (continuous) time, the herds process consists of finitely many copies of $\mathbb{T}^d$, inside which a contact process with rate λ evolves, all these processes been independent. Instead of using the epidemics terminology, we see each of these contact processes as a herd of animals (so, sites of the tree can either be empty or contain an animal/particle).



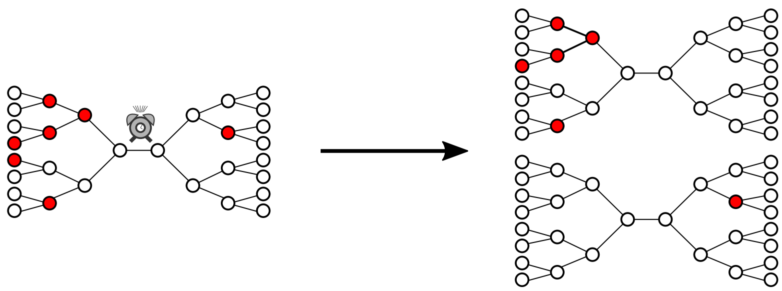
herd 1 herd 2 herd 3 herd 4 herd 5

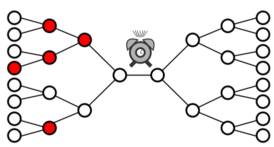
If one of these copies of $\mathbb{T}^d$ becomes empty (due to the last animal dying), then it is removed from the process. Apart from this, each edge of each copy of $\mathbb{T}^d$ has an exponential clock with rate v. The effect of such a clock ringing depends on whether or not the edge is *active* in the herd. The edge is called active in case its removal would separate some animals of the herd from others. In the following picture, active edges appear thicker and in blue, and inactive ones in black:



herd 1 herd 2 herd 3 herd 4 herd 5

If the clock of an inactive edge rings, nothing happens. If the clock of an active edge rings, the herd is split into two, as in the picture below:





No effect

This completes the description of the process. We use a formalism to describe configurations that may seem peculiar at first, but turns out to be handy to work with. We refer to a finite and non-empty set $A \subset \mathbb{T}^d$ as a *herd shape*. Then, the set of herd shapes is

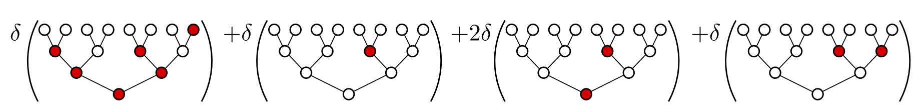
$$P_{\mathsf{f}}(\mathbb{T}^d) := \{A \subset \mathbb{T}^d :\ 0 < |A| < \infty\}.$$

The set of herd configurations is

$$\mathcal{S} := \{\xi : P_{\mathsf{f}}(\mathbb{T}^d) \to \mathbb{N}_0 \text{ with } 0 \leq \sum_A \xi(A) < \infty\}.$$

Given $\xi \in \mathcal{S}$ and $A \in P_{\mathsf{f}}(\mathbb{T}^d)$, we interpret $\xi(A)$ as the number of herds with shape A contained in ξ. For any $A \in P_{\mathsf{f}}(\mathbb{T}^d)$, we let δ_A be the element of $\mathcal{S}$ given by $\delta_A(A) = 1$ and $\delta_A(B) = 0$ for $B \neq A$ (that is, this is a configuration with a single herd, which has shape A).

A picture is worth a thousand words. If the state of the process is the one depicted in the first figure of this section, then it is encoded by $\xi \in \mathcal{S}$ given below:



(there are five herds in the configuration, but two of them have the same shape, so the corresponding δ function is multiplied by 2).

The herds process is a continuous-time Markov chain on the countable state space $\mathcal{S}$, with jump rates corresponding to the dynamics described above (the full list of birth rates is given in full in [32], but hopefully things are clear enough and they are not needed). We take the typical initial configuration $\delta_{\{o\}}$, where there is a single herd with a single animal at the origin.

We define

$$\bar{\lambda}(\mathsf{v}) := \sup\{\lambda : \text{herds process with parameters } \mathsf{v}, \lambda \text{ almost surely reaches } \xi \equiv 0\}.$$

As you may have guessed, this is the value that appears in Theorems 4.2 and 4.3. The main goal for the rest of this notes is to show the proof of the following result.

Theorem 4.4 (Schapira and V. [32]). *The function* $\mathsf{v} \mapsto \bar{\lambda}(\mathsf{v})$ *is strictly decreasing.*

4.6 Subadditivity

An *enumeration* for $\xi \in \mathcal{S}$ is a sequence $A_1, \ldots, A_n \in P_{\mathsf{f}}(\mathbb{T}^d)$ such that $\xi = \sum_{i=1}^n \delta_{A_i}$. We introduce the partial order $\preceq$ in $\mathcal{S}$ by declaring that $\xi \preceq \xi'$ if there are enumerations

$$\xi = \sum_{i=1}^m \delta_{A_i}, \quad \xi' = \sum_{i=1}^n \delta_{B_i}$$

such that $m \leq n$ and $A_i \subseteq B_i$ for $1 \leq i \leq m$.

We define $X : \mathcal{S} \to \mathbb{N}_0$ by

$$X(\xi) := \sum_A \xi(A) \cdot |A|,$$

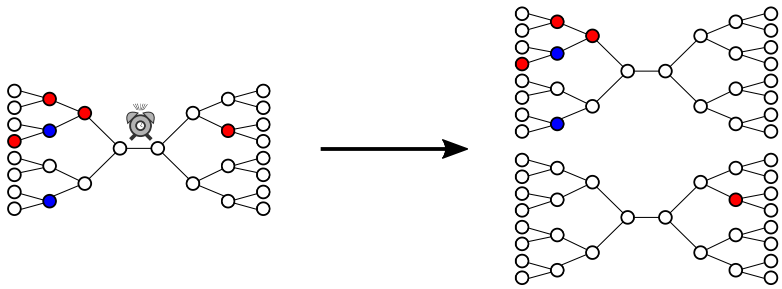
that is, $X(\xi)$ is the total number of animals in ξ.

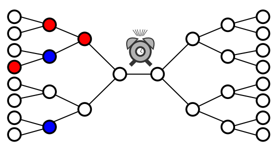
Proposition 4.6.1. *For any disjoint and non-empty* $A, B \in P_{\mathsf{f}}(\mathbb{T}^d)$*, we have*

$$\mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{A \cup B}] \leq \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_A] + \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_B]. \qquad (4.6.7)$$

To prove this, we define a *two-type herds process*:

- at any point in time, there are several copies of $\mathbb{T}^d$, and inside each of them, two independent contact processes (one *red* and one *blue*) evolve;
- edge rings work as before, and now they affect both contact processes simultaneously; in each of the examples below, a single herd is shown:





No effect

We now turn to the formalism we use to represent this process. We define

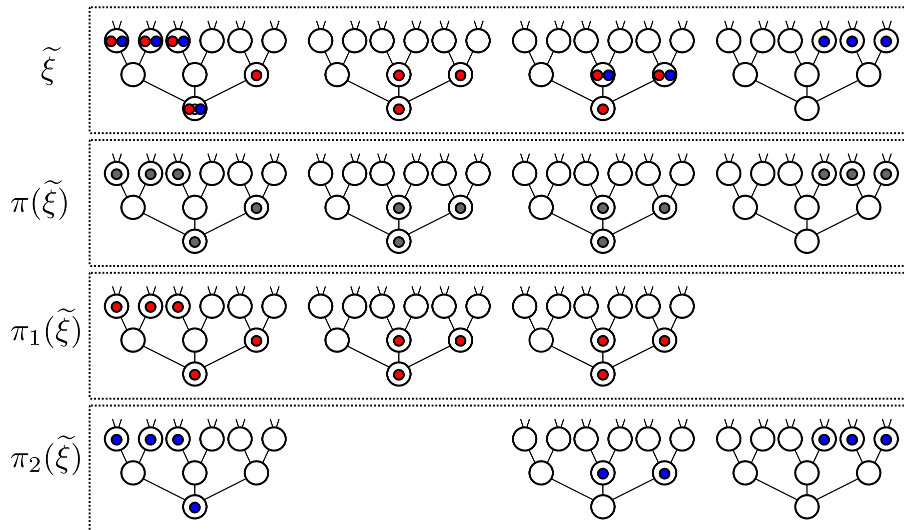
$$P_{\mathsf{f},2}(\mathbb{T}^d) := \{(A,B) : A,B \subset \mathbb{T}^d,\ A \cup B \text{ finite and non-empty}\},$$

$$\widetilde{\mathcal{S}} := \{\widetilde{\xi} : P_{\mathsf{f},2}(\mathbb{T}^d) \to \mathbb{N}_0 \text{ with } 0 < \textstyle\sum_{(A,B)} \widetilde{\xi}(A,B) < \infty\}.$$

For $\widetilde{\xi} = \sum_{i=1}^n \delta_{(A_i,B_i)} \in \widetilde{\mathcal{S}}$, we define the *projections* $\pi(\widetilde{\xi})$, $\pi_1(\widetilde{\xi})$ and $\pi_2(\widetilde{\xi}) \in \mathcal{S}$ by letting

$$\pi(\widetilde{\xi}) := \sum_{i=1}^n \delta_{(A_i \cup B_i)}, \quad \pi_1(\widetilde{\xi}) := \sum_{i : A_i \neq \varnothing} \delta_{A_i}, \quad \pi_2(\widetilde{\xi}) := \sum_{i : B_i \neq \varnothing} \delta_{B_i}.$$

For an example, see the figure below.



It will be important to observe that

$$X(\pi(\widetilde{\xi})) \leq X(\pi_1(\widetilde{\xi})) + X(\pi_2(\widetilde{\xi})). \tag{4.6.8}$$

Lemma 4.6.2. *Assume that the two-type herds process $(\widetilde{\xi}_t)_{t\geq 0}$ starts from $\widetilde{\xi}_0 = \delta_{(A,B)}$. Then,*

- $(\pi_1(\widetilde{\xi}_t))_{t\geq 0}$ *is a herds process started from δ_A;*
- $(\pi_2(\widetilde{\xi}_t))_{t\geq 0}$ *is a herds process started from δ_B;*
- $(\pi(\widetilde{\xi}_t))_{t\geq 0}$ *stochastically dominates (with respect to the partial order $\preceq$) a herds process started from $\delta_{A\cup B}$.*

This is proved by comparing jump rates of Markov chains; we skip the details.

Proof of Proposition 4.6.1. We let $(\xi_t)_{t\geq 0}$ be a herds process started from $\delta_{A\cup B}$ and $(\widetilde{\xi}_t)_{t\geq 0}$ be a two-type herds process started from $\delta_{(A,B)}$. Then,

$$\mathbb{E}[X(\xi_t)] \leq \mathbb{E}[X(\pi(\widetilde{\xi}_t))] \leq \mathbb{E}[X(\pi_1(\widetilde{\xi}_t))] + \mathbb{E}[X(\pi_2(\widetilde{\xi}_t))],$$

where the first inequality follows from Lemma 4.6.2, and the second inequality from (4.6.8). Again by Lemma 4.6.2, the right-hand side above equals the right-hand side of (4.6.7). □

Corollary 4.6.3. *For any $\xi \in \mathcal{S}$ and $t \geq 0$,*

$$\mathbb{E}[X(\xi_t) \mid \xi_0 = \xi] \leq X(\xi) \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}].$$

Proof. For any $A \in P_{\mathfrak{f}}(\mathbb{T}^d)$ with $|A| \geq 2$, fix $x \in A$ and bound

$$\mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_A] \leq \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{x\}}] + \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{A \setminus \{x\}}].$$

Note that the first probability on the right-hand side equals $\mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}]$. Iterating this, we get

$$\mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_A] \leq |A| \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}].$$

Now, given arbitrary $\xi \in \mathcal{S}$, take an enumeration $\xi = \sum_{i=1}^n \delta_{A_i}$ and bound

$$\begin{aligned}\mathbb{E}[X(\xi_t) \mid \xi_0 = \xi] &= \sum_{i=1}^n \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{A_i}] \\ &\leq \sum_{i=1}^n |A_i| \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}] = X(\xi) \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}].\end{aligned}$$

□

Corollary 4.6.4. *For any $\xi \in \mathcal{S}$ and $s, t \geq 0$,*

$$\mathbb{E}[X(\xi_{s+t}) \mid \xi_0 = \xi] \leq \mathbb{E}[X(\xi_s) \mid \xi_0 = \xi] \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}].$$

Proof. The left-hand side equals

$$\begin{aligned}&\sum_{\xi' \in \mathcal{S}} \mathbb{E}[X(\xi_{s+t} \mid \xi_s = \xi'] \cdot \mathbb{P}(\xi_s = \xi' \mid \xi_0 = \xi) \\ &\leq X(\xi') \cdot \mathbb{E}[X(\xi_{s+t}) \mid \xi_s = \delta_{\{o\}}] \cdot \mathbb{P}(\xi_s = \xi' \mid \xi_0 = \xi) \\ &= X(\xi') \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}] \cdot \mathbb{P}(\xi_s = \xi' \mid \xi_0 = \xi) \\ &= \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}] \cdot \mathbb{E}[X(\xi_s) \mid \xi_0 = \xi].\end{aligned}$$

□

Recall that **Fekete's Lemma** says that, given a function $f : [0, \infty) \to \mathbb{R}$ satisfying $f(s+t) \leq f(s) + f(t)$ for all $s, t \geq 0$, and letting $\psi := \inf_{t \geq 0} f(t) \in \mathbb{R} \cup \{-\infty\}$, we have $\lim_{t \to \infty} \frac{f(t)}{t} = \psi$.

We apply this with $f(t) := \log \mathbb{E}[X(\xi_t)]$. (Here and in what follows, if the initial condition is not explicitly stated, we assume it to be $\xi_0 = \delta_{\{o\}}$). We then get

$$\lim_{t \to \infty} \frac{f(t)}{t} = \inf_{t \geq 0} f(t) = \psi,$$

so

$$\lim_{t \to \infty} \mathbb{E}[X(\xi_t)]^{1/t} = \varphi := e^{\psi}.$$

Note that we have $f(t) \geq t \cdot \psi$ for all t, so

$$\mathbb{E}[X(\xi_t)] \geq \varphi^t \quad \text{for all } t \geq 0.$$

Why is ψ not $-\infty$? (Equivalently: why is φ not 0?). Note that

$$\mathbb{E}[X(\xi_t)] \geq \mathbb{P}(\xi_t = \delta_{\{o\}}) \geq \mathbb{P}(\text{process has no jump in } [0,t]) = e^{-(\lambda d+1)t},$$

so

$$\liminf_{t\to\infty} \mathbb{E}[X(\xi_t)]^{1/t} \geq e^{-(\lambda d+1)} > 0.$$

We sometimes write $\varphi(\lambda, \mathsf{v})$ to emphasize the dependence on λ and v.

4.7 Analysis of the growth index φ

Lemma 4.7.1. *There exists $C = C(\lambda, \mathsf{v}) > 0$ such that*

$$\mathbb{E}[X(\xi_t)] \leq C \cdot \varphi^t \quad \textit{for all } t \geq 0.$$

Proof. For $\xi \in \mathcal{S}$, define

$$Z(\xi) := \text{number of herds in } \xi \text{ containing a single particle.}$$

It is not difficult to prove that there exists $\rho = \rho(\lambda, \mathsf{v}) > 0$ such that, for any $t \geq 0$,

$$\mathbb{E}[Z(\xi_{t+1}) \mid \xi_t] \geq \rho \cdot X(\xi_t).$$

Here's a sketch of the argument. Each particle at time t can be isolated from its herd in one time unit with a probability that is bounded away from zero, by the occurrence of successive edge splits. Use this and linearity of expectation.

Next, note that

$$\mathbb{E}[X(\xi_{s+t}) \mid \xi_s] \geq Z(\xi_s) \cdot \mathbb{E}[X(\xi_t)].$$

So, for $s \geq 1$ and $t \geq 0$,

$$\begin{aligned}\mathbb{E}[X(\xi_{s+t})] = \mathbb{E}[\mathbb{E}[X(\xi_{s+t}) \mid \xi_s]] &\geq \mathbb{E}[Z(\xi_s)] \cdot \mathbb{E}[X(\xi_t)] \\ &\geq \rho \cdot \mathbb{E}[X(\xi_{s-1})] \cdot \mathbb{E}[X(\xi_s)] \\ &\geq \frac{\rho}{\mathbb{E}}[X_1] \cdot \mathbb{E}[X(\xi_s)] \cdot \mathbb{E}[X(\xi_t)].\end{aligned}$$

Iterating this, for $n \in \mathbb{N}$ we obtain

$$\mathbb{E}[X(\xi_{nt})] \geq \left(\frac{\rho}{\mathbb{E}[X_1]}\right)^{n-1} \cdot \mathbb{E}[X(\xi_t)]^n,$$

so

$$\mathbb{E}[X(\xi_t)] \leq \left(\frac{\mathbb{E}[X_1]}{\rho}\right)^{\frac{n-1}{n}} \cdot (\mathbb{E}[X(\xi_{nt})]^{\frac{1}{nt}})^t$$

for any $n \in \mathbb{N}$. This gives

$$\mathbb{E}[X(\xi_t)] \leq \frac{\mathbb{E}[X_1]}{\rho} \cdot \varphi.$$

□

Lemma 4.7.2. *The function* $(\lambda, \mathsf{v}) \mapsto \varphi(\lambda, \mathsf{v})$ *is continuous.*

Proof. Using the fact that $(\xi_t)_{t\geq 0}$ is a (non-explosive) continuous-time Markov chain on a countable state space, one can prove that $(\lambda, \mathsf{v}) \mapsto \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]$ is continuous for every t. For any $t > 0$ we have proved that

$$C(\lambda, \mathsf{v})^{-1/t} \cdot \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]^{1/t} \leq \varphi(\lambda, \mathsf{v}) \leq \mathbb{E}_{\lambda,\mathsf{v}}[\xi(X_t)]^{1/t}.$$

Then, for any $t > 0$, using the continuity of $(\lambda, \mathsf{v}) \mapsto C(\lambda, \mathsf{v})$ and of $(\lambda, \mathsf{v}) \mapsto \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]$, we bound

$$\begin{aligned}\liminf_{(\lambda',\mathsf{v}')\to(\lambda,\mathsf{v})} \varphi(\lambda', \mathsf{v}') &\geq \lim_{(\lambda',\mathsf{v}')\to(\lambda,\mathsf{v})} \left[C(\lambda', \mathsf{v}')^{-1/t} \cdot \mathbb{E}_{\lambda',\mathsf{v}'}[X(\xi_t)]^{1/t}\right] \\ &= C(\lambda, \mathsf{v})^{-1/t} \cdot \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]^{1/t}\end{aligned}$$

and

$$\limsup_{(\lambda',\mathsf{v}')\to(\lambda,\mathsf{v})} \varphi(\lambda', \mathsf{v}') \leq \lim_{(\lambda',\mathsf{v}')\to(\lambda,\mathsf{v})} \mathbb{E}_{\lambda',\mathsf{v}'}[X(\xi_t)]^{1/t} = \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]^{1/t}.$$

Finally, note that

$$\lim_{t\to\infty} \left(C(\lambda, \mathsf{v})^{-1/t} \cdot \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]^{1/t}\right) = \lim_{t\to\infty} \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]^{1/t} = \varphi(\lambda, \mathsf{v}).$$

□

Proposition 4.7.3. *For any* $\lambda > 0$, $\mathsf{v} > 0$, *the following are equivalent:*

(a) *the herds process with parameters* λ, v *survives with positive probability;*

(b) $\varphi > 1$;

(c) $\mathbb{E}[X(\xi_t)] \xrightarrow{t\to\infty} \infty$.

Proof. The inequalities $\varphi^t \leq \mathbb{E}[X(\xi_t)] \leq C\varphi^t$ give the equivalence between (b) and (c).

Assume that (a) holds. We claim that

$$\mathbb{P}\left(\lim_{t\to\infty} X(\xi_t) = \infty \mid (\xi_t) \text{ survives}\right) = 1. \tag{4.7.9}$$

To prove this, fix $r > 0$ and define the stopping times

$$\tau_0 \equiv 0, \qquad \tau_{n+1} := \inf\{t \geq \tau_n + 1 : \ X(\xi_t) \leq r\},\ n \in \mathbb{N}_0.$$

Letting $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the process, it is easy to see that

$$\text{on } \{\tau_n < \infty\}, \quad \mathbb{P}(\xi_{\tau_n+1} \text{ is dead} \mid \mathcal{F}_{\tau_n}) > \kappa(r),$$

where $\kappa(r)$ is a positive constant depending on r, λ, v. If $\tau_n < \infty$ but ξ_{τ_n+1} is dead, then $\tau_{n+1} = \infty$. This proves that

$$\text{on } \{\tau_n < \infty\}, \quad \mathbb{P}(\tau_{n+1} = \infty \mid \mathcal{F}_{\tau_n}) > \kappa(r),$$

which readily gives $\mathbb{P}(\tau_n < \infty) < (1-\kappa(r))^n$, so $\mathbb{P}(\tau_n = \infty \text{ for all } n) = 0$. This implies that

$$\mathbb{P}\left((\xi_t) \text{ survives but } \liminf_{t\to\infty} X(\xi_t) \le r\right) \le \mathbb{P}(\tau_n < \infty \text{ for all } n) = 0.$$

Since r is arbitrary, we readily obtain (4.7.9). Now, by Fatou's Lemma,

$$\liminf_{t\to\infty} \mathbb{E}[X(\xi_t)] \ge \mathbb{E}\left[\liminf_{t\to\infty} X(\xi_t)\right] \ge \infty \cdot \mathbb{P}\left(X(\xi_t) \xrightarrow{t\to\infty} \infty\right) = \infty,$$

so (c) holds.

Now assume that (c) holds. Recalling the definition of $Z(\xi)$ and the inequality $\mathbb{E}[Z(\xi_t)] \ge \rho\mathbb{E}[X(\xi_{t-1}]$ from earlier, there exists $t \ge 0$ such that $\mathbb{E}[Z(\xi_t)] > 1$. That is, at time t, the expected number of herds containing a single particle is larger than 1. Each of these herds can then be seen as the starting point of a fresh herds process, which, when allowed to evolve for extra t time units, produces another random number (with expectation above 1) of herds with a single particle etc. This produces a supercritical branching process embedded inside the herds process. Clearly, if this branching process survives, then so does the herds process. This shows that (a) holds. □

The following is a direct consequence of Lemma 4.7.3 and Proposition 4.7.2.

Corollary 4.7.4. *For any* v, *we have* $\varphi(\bar{\lambda}(\mathsf{v}), \mathsf{v}) = 1$.

Our main monotonicity result, to be discussed in detail in the next section, is the following.

Proposition 4.7.5. *The function* $(\lambda, \mathsf{v}) \mapsto \varphi(\lambda, \mathsf{v})$ *is strictly increasing in both arguments.*

We are now in a position to conclude the proof of our main theorem.

Proof of Theorem 4.4. Let $\mathsf{v}' > \mathsf{v} > 0$. We have

$$\varphi(\bar{\lambda}(\mathsf{v}'), \mathsf{v}') = 1 = \varphi(\bar{\lambda}(\mathsf{v}), \mathsf{v}) < \varphi(\bar{\lambda}(\mathsf{v}), \mathsf{v}').$$

Again using the strict monotonicity of φ, we conclude from $\varphi(\bar{\lambda}(\mathsf{v}'), \mathsf{v}') < \varphi(\bar{\lambda}(\mathsf{v}), \mathsf{v}')$ that $\bar{\lambda}(\mathsf{v}') < \bar{\lambda}(\mathsf{v})$. □

4.8 Monotonicity

In this section, we discuss some of the ideas that go into the proof of Proposition 4.7.5. We will concentrate on the monotonicity of φ with respect to v; the monotonicity with respect to λ can be handled in a very similar manner. As we will see, the starting point is the study of the monotonicity of the simpler function $(\lambda, \mathsf{v}) \mapsto \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t)]$ for fixed $t \ge 0$.

Some more definitions will be necessary. Given $A \in P_{\mathsf{f}}(\mathbb{T}^d)$, we say that an edge e of $\mathbb{T}^d$ is *active* for A if A intersects both the components of $\mathbb{T}^d$ that arise from the removal of e. The intersections of A with these two components are then denoted $A_{e,1}$ and $A_{e,2}$ (with 1 and 2 assigned in some arbitrary way).

We then define

$$g_{\lambda,\mathsf{v}}(A,t) = \mathsf{v}\cdot \sum_{e \text{ active of } A} \left(\mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t) \mid \xi_0 = \delta_{A_{e,1}} + \delta_{A_{e,2}}] - \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t) \mid \xi_0 = \delta_A]\right).$$

Note that, by Proposition 4.6.1, we have $g_{\lambda,\mathsf{v}}(A,t) \geq 0$. Also define, for $\xi \in \mathcal{S}$,

$$g_{\lambda,\mathsf{v}}(\xi,t) = \sum_A \xi(A) \cdot g_{\lambda,\mathsf{v}}(A,t).$$

The quantity $g_{\lambda,\mathsf{v}}(\xi,t)$ gives the sum of the effects (multiplied by the rate v) on the expected number of particles at time t of splitting a single edge in a single herd of the initial configuration.

The proof of Proposition 4.7.5 needs two ingredients, Lemma 4.8.1 and Lemma 4.8.2 below. We will not reproduce their proofs (the first one is quite long, and the second a bit clumsy to write), but in each case we provide an heuristic explanation.

Lemma 4.8.1. *For any $T > 0$ we have*

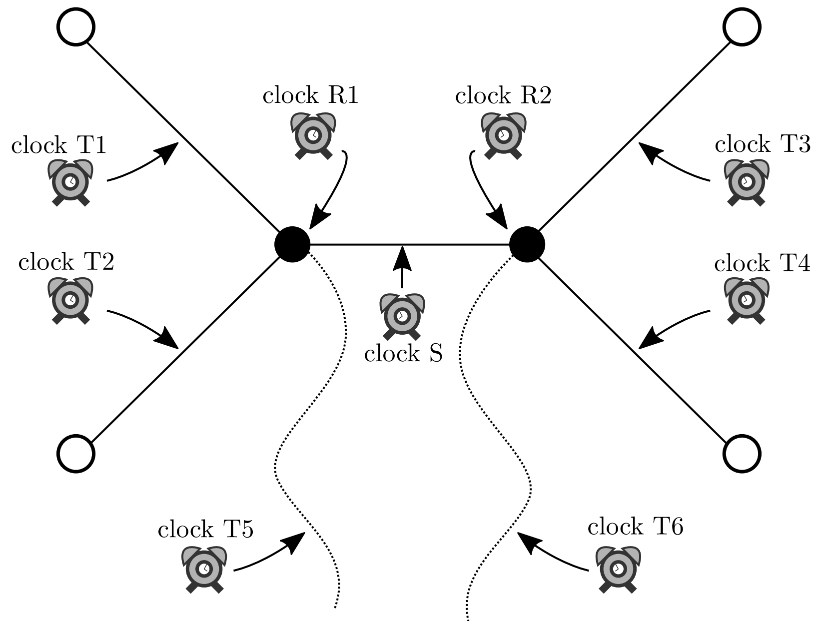
$$\frac{\partial}{\partial \mathsf{v}}\mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_T) \mid \xi_0 = \delta_{\{o\}}] = \int_0^T g_{\lambda,\mathsf{v}}(\xi_t, T-t)\,\mathrm{d}t. \tag{4.8.10}$$

Heuristics of proof. Suppose that we increase v by a tiny amount, to $\mathsf{v}+\epsilon$. The chains with parameters (λ,v) and $(\lambda,\mathsf{v}+\epsilon)$ have very similar jump rates; we run them coupled together until (and if) there is a moment when the second chain attempts an ϵ-jump, that is, a jump that is caused by the incremental splitting rate. At this point, we allow the jump of the second chain (the first chain does nothing), and from then onwards, the two chains evolve independently. This coupling is considered in the time interval $[0,T]$. When ϵ is tiny (depending on T), a single ϵ-jump captures the larger-order part of the discrepancy between $\mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_T)]$ and $\mathbb{E}_{\lambda,\mathsf{v}+\epsilon}[X(\xi_T)]$: having more than one ϵ-jump in $[0,T]$ has probability of order $O(\epsilon^2)$. The integral on the right-hand side of (4.8.10) involves the instant in $[0,T]$ at which the single-jump discrepancy may occur, and the function $g_{\lambda,\mathsf{v}}(\xi_t, T-t)$ contains the rate and the effect of such a jump. □

Lemma 4.8.2. *There exists $\gamma > 0$ depending continuously on λ and v such that the following holds. Let v be a vertex of $\mathbb{T}^d$ neighboring the root. Then, for any $t > 1$,*

$$\mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}} + \delta_{\{v\}}] \geq \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o,v\}}] + \gamma \cdot \mathbb{E}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}].$$

Heuristics of proof. The proof involves a coupling between the processes with the two initial configurations $\delta_{\{o,v\}}$ and $\delta_{\{o\}} + \delta_{\{v\}}$. We refer to the process started from $\delta_{\{o,v\}}$ as the *friendly* process, and to the process started from $\delta_{\{o\}}+\delta_{\{v\}}$ as the *unfriendly* one. To explain the coupling, we refer to the figure below.



There are several exponential clocks: two recovery clocks with rate 1, six transmission clocks with rate λ, and one splitting clock with rate v. If none of the clocks rings before time 1, then none of the two processes does anything in $[0,1]$, and we allow them to evolve independently afterwards.

Now assume that one of the clocks rings at some time $t^* < 1$. We leave the two processes quiet before t^*, and the decision of what to do at time t^* depends on which clock is ringing. Regardless of what we do, and what the resulting configurations at t^* are, the two processes continue evolving independently after that.

The rules for time t^* are:

(a) if the clock ringing is R1, R2, T1, T2, T3, or T4, we are able to perform the corresponding operation (a transmission or a recovery) in both the friendly and the unfriendly process simultaneously;

(b) if the clock ringing is S, we perform the edge split in the unfriendly process, and do nothing for the friendly one;

(c) if the clock ringing is T5 or T6, then we perform a transmission and gain a particle in the unfriendly process, but (crucially) we do nothing in the friendly one.

Now, in all scenarios, the resulting configuration in the unfriendly process is at least as powerful (in light of Proposition 4.6.1) as the one in the friendly process. Moreover, in scenario (c), it is *strictly more powerful*, because in this scenario, at time t^*, the unfriendly process has a herd with two neighboring particles (same as the friendly one), plus an extra herd with a single particle. This extra herd is the starting entity in an independent herds process from then onwards, and this is what yields the strict inequality in the lemma (γ has to do with the probability of scenario (c)).

□

We are now ready to put the two ingredients together and prove Proposition 4.7.5.

Proof of Proposition 4.7.5. Given $\xi \in \mathcal{S}$, let $Y(\xi)$ denote the number of herds in ξ that contain exactly two particles, and so that these particles are in neighboring positions. By Lemma 4.8.2, for any $t \geq 1$,

$$g_{\lambda,\mathsf{v}}(\xi, t) \geq \kappa_1(\lambda, \mathsf{v}) \cdot Y(\xi) \cdot \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}]$$

for some $\kappa_1(\lambda, \mathsf{v}) > 0$. For any $t \geq 1$,

$$\mathbb{E}[Y(\xi_t) \mid \mathcal{F}_{t-1}] \geq \kappa_2(\lambda, \mathsf{v}) \cdot X(\xi_{t-1})$$

for some $\kappa_2(\lambda, \mathsf{v}) > 0$. The proof of this is somewhat clumsy to write, but easy to understand. Each particle alive at time $t-1$ has a chance, in the following time unit, of first isolating itself from all other particles (through successive edge splits), and then producing another particle at a neighboring position; the inequality is proved from this and linearity of expectation.

Define

$$F(\lambda, \mathsf{v}, t) := \mathbb{E}_{\lambda,\mathsf{v}}[X(\xi_t) \mid \xi_0 = \delta_{\{o\}}], \quad t \geq 0.$$

By combining the above inequalities similarly to what was done in the proof of Lemma 4.7.1, we can prove that

$$\mathbb{E}_{\lambda,\mathsf{v}}[g_{\lambda,\mathsf{v}}(\xi_t, T-t)] \geq \kappa_3(\lambda, \mathsf{v}) \cdot F(\lambda, \mathsf{v}, t) \cdot F(\lambda, \mathsf{v}, T-t),$$

so, using Corollary 4.6.4, we obtain

$$\mathbb{E}_{\lambda,\mathsf{v}}[g_{\lambda,\mathsf{v}}(\xi_t, T-t)] \geq \kappa_3(\lambda, \mathsf{v}) \cdot F(\lambda, \mathsf{v}, T).$$

Now, Lemma 4.8.1 gives

$$\frac{\partial F}{\partial \mathsf{v}}(\lambda, \mathsf{v}, T) \geq \kappa_3(\lambda, \mathsf{v}) \cdot T \cdot F(\lambda, \mathsf{v}, T),$$

so

$$\frac{\partial}{\partial \mathsf{v}} \log F(\lambda, \mathsf{v}, T) \geq \kappa_3(\lambda, \mathsf{v}) \cdot T.$$

Given $\mathsf{v}_1 < \mathsf{v}_2$, integrating the above on $[\mathsf{v}_1, \mathsf{v}_2]$ gives

$$\log \frac{F(\lambda, \mathsf{v}_2, T)}{F(\lambda, \mathsf{v}_1, T)} \geq \kappa_* \cdot T \cdot (\mathsf{v}_2 - \mathsf{v}_1),$$

where $\kappa_* := \min_{\mathsf{v}\in[\mathsf{v}_1,\mathsf{v}_2]} \kappa_3(\lambda,\mathsf{v})$. Going over the different constants that were involved in the choice of κ_3, it is straightforward to see that κ_3 depends continuously on λ and v, which implies that $\kappa_* > 0$. We now have

$$\frac{F(\lambda,\mathsf{v}_2,T)^{1/T}}{F(\lambda,\mathsf{v}_1,T)^{1/T}} \geq e^{\kappa_*\cdot(\mathsf{v}_2-\mathsf{v}_1)}.$$

Taking $T\to\infty$ gives

$$\frac{\varphi(\lambda,\mathsf{v}_2)}{\varphi(\lambda,\mathsf{v}_1)} \geq e^{\kappa_*\cdot(\mathsf{v}_2-\mathsf{v}_1)} > 1.$$

□

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