

## Category Theory

### Exercise Sheet 12

**Exercise 1.** Consider functors  $F: C \rightarrow D$  and  $G: D \rightarrow C$ .

- (i) Consider the natural transformations  $\eta: \text{id}_C \rightarrow GF$  and  $\epsilon: FG \rightarrow \text{id}_D$  associated to an adjunction between  $F$  and  $G$ . Show that the following diagrams of natural transformations commute:

$$\begin{array}{ccc}
 F & \xrightarrow{F\eta} & FGF \\
 & \searrow \text{id}_F & \downarrow \epsilon F \\
 & & F
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xrightarrow{\eta G} & GFG \\
 & \searrow \text{id}_G & \downarrow G\epsilon \\
 & & G
 \end{array}$$

- (ii) Inversely, from such natural transformations  $\eta$  and  $\epsilon$  for which the above triangles commute, construct an adjunction between  $F$  and  $G$ .

**Exercise 2.** Consider the forgetful functor from the category of partially ordered sets to the category of sets. Show that this functor has a left adjoint and describe the associated monad  $T: \text{Set} \rightarrow \text{Set}$  as well as the category  $\text{Set}^T$  of  $T$ -algebras.

**Exercise 3.** Let  $T$  be a monad on a category  $C$ . Show that for any index category  $I$ , if  $I$ -indexed limits exist in  $C$ , then they exist in  $C^T$ .