Category Theory
Exercise Sheet 4

Lecture Homepage: https://www.math.cit.tum.de/algebra/lehre/sommersemester-2022/ss2022-category-theory/

Exercise 1. Consider the following properties of functors: faithful, full, essentially surjective. For each combination of these properties, find a functor having exactly these properties.

Exercise 2. Show that the composition of two equivalences of categories is again an equivalence.

Exercise 3. Let C be the category of finitely generated abelian groups.

(1) Show that for each $n \in \mathbb{Z}$, there is a natural transformation $\alpha_n \colon \mathrm{id}_C \to \mathrm{id}_C$ which for every $A \in C$ is given by the homomorphism $A \to A, a \mapsto na$.
(2) Show that every natural transformation $\mathrm{id}_C \to \mathrm{id}_C$ is equal to α_n for some $n \in \mathbb{Z}$.
(3) For $A \in C$, let $TA \subset A$ be the torsion subgroup of A. We consider the functor

$$\begin{aligned} G \colon C &\to C \\ A &\mapsto TA \oplus A/TA. \end{aligned}$$

By the structure theorem for finitely generated abelian groups, for each $A \in C$ there exists an isomorphism

$$A \cong TA \oplus A/TA.$$

Show that there is no way to choose such isomorphisms for all $A \in C$ in such a way that they form an isomorphism $\mathrm{id}_C \cong G$ of functors.