Technische Universität München
Zentrum Mathematik

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Sheet 3

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Introduction to Algebraic Number Theory

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Exercise 1. Show that $x^2 - 10y^2 = \pm 2$ has no solutions in $\mathbb{Z}$. Deduce that if α divides both 2 and $\sqrt{10}$ in $\mathbb{Z}[\sqrt{10}]$, then α is a unit. Show however that one cannot write $1 = 2\beta + \sqrt{10}\gamma$ with $\beta, \gamma \in \mathbb{Z}[\sqrt{10}]$.

Exercise 2. Let $K = \mathbb{Q}(\alpha)$ be a number field of degree $n = [K : \mathbb{Q}]$ and $f \in \mathbb{Z}[X]$ the minimal polynomial of α. Show that

$$d(1, \alpha, \ldots, \alpha^{n-1}) = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbb{Q}}(f'(\alpha)).$$

Exercise 3. Prove Stickelberger's Theorem: The discriminant d_K of a number field K satisfies

$$d_K \equiv 0, 1 \pmod 4.$$

Hint: For an integral basis $v_1, \ldots, v_n$ of $\mathcal{O}_K$ consider the expansion of the determinant of $(\sigma_i(v_j))_{ij}$ as a sum over permutations of $\{1, \ldots, n\}$. Write P for the sum of those terms which occur with positive sign and N for the sum of those terms which occur with negative sign, so that $\det(\sigma_i(v_j)) = P - N$. Show that $P + N$ and PN are both invariant by $\mathrm{Gal}(K/\mathbb{Q})$ and hence are in $\mathbb{Z}$.