Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 8

# Abelian varieties (MA 5115)

As usual, we work over an algebraically closed field $k$.
We had seen in the lecture that for any finite subgroup $G \subset A(k)$ of an Abelian variety, the geometric quotient $A \twoheadrightarrow A /\!\!/ G$ is an isogeny with (schematic) kernel $G$. In the first exercise, we are proving the converse, i.e that if $\varphi\colon A \to A'$ is an isogeny of Abelian varieties with $(\ker\varphi)_{sch} = G$, a finite algebraic group, then $A' \cong A /\!\!/ G$. Then, in the second exercise, we prove that these are precisely the étale isogenies.

**Exercise 1.** Let $A$ be an Abelian variety and $G \subset A$ be a finite algebraic group (i.e. a finite closed subgroupvariety).

(a) As preparation show the that the degree of an isogeny is multiplicative: Let $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A''$ be isogenies. Prove that $\varphi' \circ \varphi$ is an isogeny of degree

$$\deg(\varphi' \circ \varphi) = \deg\varphi' \cdot \deg\varphi.$$

(b) Now let $\varphi\colon A \to A'$ be a morphism with $(\ker\varphi)_{sch} = G$. Show that there exists a unique isogeny $\varphi'\colon A /\!\!/ G \to A'$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ {\scriptstyle\pi}\downarrow & \nearrow_{\varphi'} & \\ A /\!\!/ G & & \end{array}$$

commutes. Moreover $\deg\varphi' = 1$.

(c) Prove that an isogeny is an isomorphism if and only if it has degree 1. Deduce that there exists a (unique) isomorphism $A /\!\!/ G \cong A'$, which identifies $\varphi$ and $\pi$.

**Exercise 2.** Before we prove the main statement, we need two results about tangent spaces of algebraic groups.

(a) Let $G$ be a finite group *scheme* over $\operatorname{Spec} k$. Show that $G$ is an algebraic group (i.e. reduced) if and only if $T_e G = \{0\}$.

(b) Let $\varphi\colon H \to H'$ be a morphism of algebraic groups with $(\ker\varphi)_{sch} = K$. Show that

$$\ker((d\varphi)_e\colon T_e H \to T_e H') = T_e K.$$

(c) Deduce from (a) and (b) that an isogeny $\varphi\colon A \to A'$ of Abelian varieties is étale if and only if $(\ker\varphi)_{sch}$ is reduced.

Deadline: Monday, 25th January, 2021