winter semester 2019/20 Claudia Scheimbauer

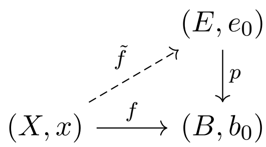
Lifting of covering maps

Convention. We write $f\colon (X, x) \to (Y, y)$ when $f\colon X \to Y$ is continuous and $f(x) = y$. This is called a *pointed map.*

Warning. The book [1] always assumes covering maps to be continuous maps between spaces which are path-connected and locally path-connected.

The following proposition is Proposition 1.33 in [2] and Lemma 80.2 in [1].

Proposition. *Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, X a path-connected and locally path-connected space, and $f\colon (X, x) \to (B, b_0)$ a continuous map. Then, a lift $\tilde{f}\colon (X, x) \to (B, b_0)$ exists if and only if $f_*(\pi_1(X, x)) \subseteq p_*(\pi_1(E, e_0))$. If this is the case, then the lift $\tilde{f}$ is unique.*



Furthermore, if f is a covering map and B is locally path-connected and then $\tilde{f}$ is a covering map.

Proof. If a lift $\tilde{f}$ exists, then $p \circ \tilde{f} = f$, so $f_* = p_*\tilde{f}_*$, which implies the "only if" statement.

For the converse direction, our strategy is the following. Note that if $(X, x) = ([0, 1], 0)$, then the statement follows from the path lifting lemma. In the general case we use path lifting to define a candidate for $\tilde{f}$.

Let $y \in X$. Then, since X is path-connected, we can find a path γ_y from x to y. Composing with f we obtain a path $f \circ \gamma_y$ in B starting at $f(x) = b_0$. By the path lifting lemma, there is a unique lift $\widetilde{f\gamma_y}$ of $f \circ \gamma_y$ with $\widetilde{f\gamma_y}(0) = e_0$. Define

$$\tilde{f}(y) \coloneqq \widetilde{f\gamma_y}(1).$$

We need to check that

(1) $\tilde{f}$ is well-defined,

(2) $\tilde{f}$ is continuous,

(3) $\tilde{f}$ is unique, and

(4) if f is a covering map, then $\tilde{f}$ also is.

For (1), we will use the homotopy lifting theorem. We need to show that the definition of $\tilde{f}$ was independent of the choice of the path γ_y. Let α_y be another path from x to y. Then we can concatenate the images of α_y and γ_y^{-1} under f to obtain a loop $\delta = f\alpha_y * f\gamma_y^{-1}$ in B based at b_0. Then

$$[\delta] = [f \circ (\alpha_y * \gamma_y^{-1})] = f_*[\alpha_y * \gamma_y^{-1}] \in f_*\pi_1(X, x) \subseteq p_*\pi_1(E, e_0),$$

so there is a loop ε in E based at e_0 such that $[\delta] = p_*[\varepsilon] = [p \circ \varepsilon]$. This in turn implies that there is a path homotopy H in B from δ to $p \circ \varepsilon$. Note that $p \circ \varepsilon$ lifts to the loop ε in E based at e_0. By the homotopy lifting theorem, there is a lifted homotopy $\tilde{H}$ in E such that $\tilde{H}_1 = \varepsilon$, which is a loop. Therefore $\tilde{H}_0$ also is a loop and $p \circ \tilde{H}_0 = \delta$. By the uniqueness of path lifting we have that in $[0, \frac{1}{2}]$ the path $\tilde{H}_0$ runs through $\widetilde{f\alpha_y}$ and in $[\frac{1}{2}, 1]$ the path H_0 runs through $\widetilde{f\gamma_y^{-1}}$. In particular, at the midpoint $\frac{1}{2}$ they agree, so

$$\widetilde{f\alpha_y}(1) = \widetilde{f\gamma_y}(1).$$

For (2), we will show that for any $y \in X$ and $\tilde{U}$ an open neighborhood of $\tilde{f}(y)$ the inverse image $\tilde{f}^{-1}(\tilde{U})$ contains an open neighborhood V of y. We will use:

(a) Since p is a local homeomorphism, we can assume (by perhaps shrinking $\tilde{U}$) that $\tilde{f}|_{\tilde{U}}$ is a homeomorphism onto $U := p(\tilde{U})$, which is open in B.

(b) Since f is continuous, $f^{-1}(U)$ is open and contains y. By the local path-connectedness of X, there is a path-connected open neighborhood V of y with $V \subseteq f^{-1}(U)$.

We claim that $\tilde{f}(V) \subseteq \tilde{U}$. To see this, fix a path γ_y from x to y. For any $y' \in V$, since v is path-connected, we can pick a path $\eta_{y'}$ from y to y' in V. Then $\gamma_{y'} = \gamma_y * \eta_{y'}$ is a path from x to y', so we can use it to evaluate $\tilde{f}$ at y'. We pick a lift $\widetilde{f \circ \gamma_y}$ of $f \circ \gamma_y$ starting at e_0. A lift of $f \circ \eta_{y'}$ starting at $\widetilde{f \circ \gamma_y}(1)$ is given by $(p|_{\tilde{U}})^{-1} \circ f \circ \eta_{y'}$ because $p|_{\tilde{U}}$ is a homeomorphism. So

$$\tilde{f}(y') = \widetilde{f \circ \eta_y}(1) = (p|_{\tilde{U}})^{-1} \circ f \circ \eta_{y'}(1) \in \tilde{U}.$$

You can also see directly that $\tilde{f}|_V = (p|_{\tilde{U}})^{-1} \circ f$, so $\tilde{f}$ is continuous in V, and the collection of such V's cover X.

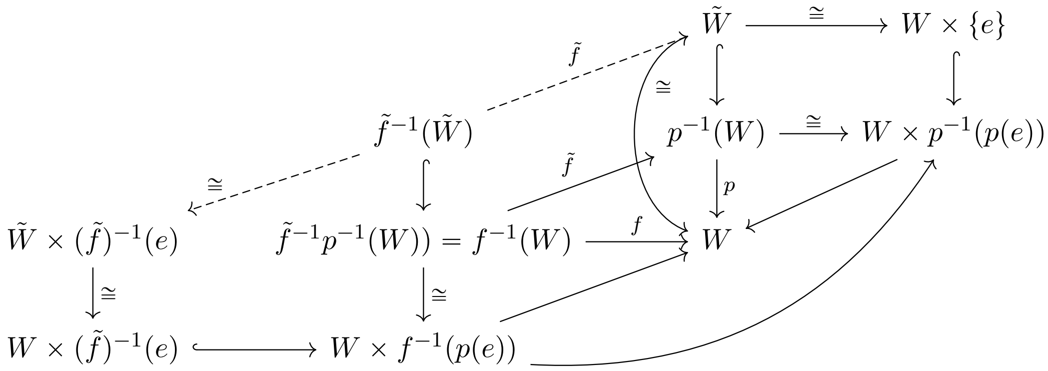
For (3), the uniqueness follows from the uniqueness of lifts of paths.

For (4), let $e \in E$. Consider the open set $\tilde{U}$ from (a). Then $p(e) \in B$, so since f is a covering space, there is a trivializing open neighborhood V in B such that

$$\begin{array}{ccc} f^{-1}(V) & \xrightarrow{\cong} & V \times f^{-1}(p(e)) \\ \big\downarrow{\scriptstyle f|_V} & \swarrow{\scriptstyle pr_1} & \\ V & & \end{array}$$

commutes. Set $W := V \cap p(\tilde{U})$. We further assume (by possibly shrinking) that W is path-connected. Then $(\tilde{f})^{-1}(p^{-1}(W)) = f^{-1}(W)$ and we claim that $\tilde{W} := p^{-1}(W) \cap \tilde{U}$ is a trivializing open neighborhood for $\tilde{f}$. Note that p restricts to a homeomorphism from $\tilde{W}$ onto

W. The claim follows from chasing through details in the following diagram:

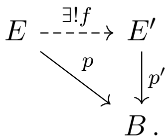


The two dashed arrows are part of the trivializing diagram for $\tilde{f}$, and the third one is the composition of: the lowest horizontal arrow, the curved lower right arrow, the projection onto W and then the curved vertical arrow to $\tilde{W}$. □

Definition. A covering space $p : E \to B$ for which E is simply connected is called a *universal cover* of B.

However, it is only “universal” (meaning that it has a universal property), if B is locally path-connected and path connected.

Corollary. *Let $p : E \to B$ be a universal cover of a locally path-connected and path-connected base B. Then if $p' : E' \to B$ is any other cover, where E' is path-connected, then there is a unique map of covering spaces $f : E \to E'$,*



In particular, the universal covering is unique up to isomorphism of covering spaces.

We also need to add “locally path-connected” to the Theorem relating deck transformations and the fundamental group.

Theorem. *Let $p : E \to B$ be a universal covering, where E is locally path-connected, and let $p(e_0) = b_0$. Then*

$$\mathrm{Aut}(p) \cong \pi_1(B, b_0).$$

References

[1] James R. Munkres. *Topology.* Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

[2] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. available at `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`.